forallχ: SFU

An introduction to formal logic

P.D. Magnus
University at Albany, State University of New York

Thomas Donaldson
Simon Fraser University

Bruno Guindon
Simon Fraser University

P.D. Magnus would like to thank the people who made this project possible. Notable among these are Cristyn Magnus, who read many early drafts; Aaron Schiller, who was an early adopter and provided considerable, helpful feedback; and Bin Kang, Craig Erb, Nathan Carter, Wes McMichael, Selva Samuel, Dave Krueger, Brandon Lee, Toan Tran, Marcus Adams, Matthew Brown, and the students of Introduction to Logic, who detected various errors in previous versions of the book.

Thomas Donaldson and Bruno Guindon, who modified the original version of `forall`χ, would like to thank the original author P.D. Magnus for his act of generosity in making `forall`χ available to everyone. Donaldson and Guindon would especially like to thank Kesavan Thanagopal and Celia Gentle for their valuable feedback, meticulous proofreading, and expert typesetting in the development of `forall`χ: SFU. A very special mention goes out to Celia for drafting up most, and typesetting all, solutions to the exercises. Donaldson and Guindon would also like to thank Transforming Inquiry into Teaching + Learning (TILT) at SFU for awarding them a grant that made this project possible.

Typesetting was carried out entirely in LaTeX2ε. The style for typesetting proofs is based on fitch.sty (v0.4) by Peter Selinger, University of Ottawa.

This copy of `forall`χ: SFU is current as of January 12, 2026.

`forall`χ: SFU **contents**

II First-order logic 88

III Solutions 156

Chapter 1

What is logic?

Logic is the business of evaluating arguments, sorting good ones from bad ones. In everyday language, we sometimes use the word ‘argument’ to refer to belligerent shouting matches. If you and a friend have an argument in this sense, things are not going well between the two of you.

In logic, we are not interested in the teeth-gnashing, hair-pulling kind of argument. A logical argument is structured to give someone a reason to believe some conclusion. Here is one such argument:

(1) It is raining heavily.

(2) If it’s raining and you do not take an umbrella, you will get soaked.

∴ You should take an umbrella.

The three dots on the third line of the argument mean ‘Therefore’ and they indicate that the final sentence is the *conclusion* of the argument. The other sentences are *premises* of the argument. If you believe the premises and the argument is a good one, then the argument provides you with a reason to believe the conclusion.

In this chapter, we will begin a discussion of the *structure* of arguments. We will see that an argument is composed of one or more *inferences*, and that an inference is composed of *statements*. We will also discuss some basic logical notions.

1.1 Statements

In logic, we are only interested in sentences that can appear in arguments. So we will say that a STATEMENT is a sentence (or part of a sentence) that is either true or false.

Questions 'Are you sleepy yet?' is a sentence, but it is not a statement. Suppose you answer the question, 'I am not sleepy.' This is either true or false, and so it is a statement. Generally, *questions* will not count as statements, but *answers* will.

Imperatives Commands are often phrased as imperatives like 'Wake up!', 'Sit up straight', and so on. These are sentences, but they are not statements.

Exclamations 'Ouch!' is not a statement, because it is neither true nor false. We will treat 'Ouch, I hurt my toe!' as meaning the same thing as 'I hurt my toe.' The 'ouch' does not add anything that could be true or false.

You should not confuse the distinction between statements and other sentences with the distinction between fact and opinion. Often, sentences in logic will express things that would count as facts—such as 'Kierkegaard was a hunchback' or 'Kierkegaard liked almonds.' They can also express things that you might think of as matters of opinion—such as, 'Almonds are yummy.'

We say that a statement is a sentence *'or part of a sentence'* that can be true or false, because it often happens that statements can be combined to make larger statements. Consider, for example, the following statement:

Vancouver is in British Columbia and British Columbia is in Canada.

In this example, the two statements 'Vancouver is in British Columbia' and 'British Columbia is in Canada' have been combined using the word 'and' to make a single, larger statement. This point—that statements can be joined together to make bigger statements—will be very important later on in the book.

1.2 Inferences

We can define an INFERENCE as a series of statements. The statements at the beginning of the series are premises. The final statement in the series is the conclusion. The inference is supposed to *justify* the conclusion, using the premises as starting points.

When people give inferences, they often flag the premises using phrases like 'since', 'because', and 'given that'. And they often flag the conclusion using phrases like 'therefore' and 'so'. Words like these are a clue to the structure of the writer's line of reasoning, especially if—in the argument as given—the conclusion comes at the beginning or in the middle of the argument.

premise indicators: since, because, given that

conclusion indicators: therefore, hence, thus, then, so

1.3 Evaluating inferences

Consider the inference that you should take an umbrella (on p. 6, above). If premise (1) is false—if it is sunny outside—then the inference gives you no reason to carry an umbrella.

Premise (2) might also be false. Perhaps you wear a rain poncho that keeps you dry even when you walk in the rain without an umbrella. Perhaps you will keep to covered walkways.

But suppose for a moment that both the premises *are* true. It is raining. You do not own a rain poncho. You need to go places where there are no covered walkways. Now does the inference show you that you should take an umbrella? Not necessarily. Perhaps you enjoy walking in the rain, and you would like to get soaked. In that case, even though the premises were true, the conclusion would be false.

For any inference, there are two ways that it could be weak. First, one or more of the premises might be false. An inference gives you a reason to believe its conclusion only if you believe its premises. Second, the premises might not be connected to the conclusion in the right way: it might be that, *even supposing the premises to be true*, they don't support the conclusion.

Consider another example:

You are reading this book.
This is a logic book.
∴ You are a logic student.

This is not a terrible inference. Most people who read this book are logic students. Yet, it is possible for someone besides a logic student to read this book. If your roommate picked up the book and thumbed through it, they would not immediately become a logic student. So the premises of this inference, even though they are true, do not guarantee the truth of the conclusion. Its logical form is less than perfect.

An inference that has no weakness of the second kind would have perfect logical form. If its premises were true, then its conclusion would *necessarily* be true. We call such an inference 'deductively valid' or just 'valid.'

Even though we might count the inference above as a good argument in some sense, it is not valid; that is, it is 'invalid'. One important task of logic is to sort valid inferences from invalid inferences.

It is important, when studying logic, to remember that the word 'valid' is used in this very specific way. It is not just an unspecific term of approval!

1.4 Deductive validity

An inference is DEDUCTIVELY VALID, or just VALID for short, if and only if there is no possible situation in which the premises are true and the conclusion is false.

The crucial thing about a valid inference is that it is impossible for the premises to be true *at the same time* that the conclusion is false. Consider this example:

Oranges are either fruits or musical instruments.
Oranges are not fruits.
∴ Oranges are musical instruments.

The conclusion of this inference is ridiculous. Nevertheless, it follows validly from the premises. This is a valid argument. *If* both premises were true, *then* the conclusion would necessarily be true.

This shows that a deductively valid inference does not need to have true premises or a true conclusion. Conversely, having true premises and a true conclusion is not enough to make an inference valid.

Consider this example:

London is in England.
Beijing is in China.
∴ Paris is in France.

The premises and conclusion of this inference are, as a matter of fact, all true. This is a terrible inference, however, because the premises have nothing to do with the conclusion. Imagine what would happen if Paris declared independence from the rest of France. Then the conclusion would be false, even though the premises would both still be true. Thus, it is *logically possible* for the premises of this inference to be true and the conclusion false. The argument is invalid.

The important thing to remember is that validity is not about the actual truth or falsity of the statements in the inference. Instead, it is about the form of the argument: The truth of the premises is incompatible with the falsity of the conclusion.

Inductive inferences

There can be good arguments which nevertheless fail to be deductively valid. Consider this one:

In January 1997, it rained in San Diego.
In January 1998, it rained in San Diego.
In January 1999, it rained in San Diego.
∴ It rains every January in San Diego.

This is an INDUCTIVE inference, because it generalizes from many cases to a conclusion about all cases.

Certainly, the inference could be made stronger by adding additional premises: In January 2000, it rained in San Diego. In January 2001 . . . and so on. Regardless of how many premises we add, however, the inference will still not be deductively valid. It is possible, although unlikely, that it will fail to rain next January in San Diego. Moreover, we know that the weather can be fickle. No amount of evidence should convince us that it rains there *every* January. Who is to say that some year will not be a freakish year in which there is no rain in January in San Diego? All it takes to make the conclusion of the inference false is just one instance of a rainless January.

Inductive inferences, even good ones, are not deductively valid. We will not be interested in inductive inferences in this book.

Abductive inferences

Another kind of arguments that can be good but are not deductively valid are ABDUCTIVE arguments, or inferences to the best explanation. Consider the following inference:

The human eye is incredibly complex.
The only possible explanation for this level of complexity is that the human eye was created by a supernatural divine creator.
∴ God must exist.

The way abductive arguments work is by first starting with some putative phenomenon that requires explanation—in this case, the complexity of the human eye—and then offering what is taken to be the best explanation of that phenomenon—in this case, that it is the product of a supernatural creator. The idea is that if the best explanation for some phenomenon or observation requires us to posit certain things, then we are justified to infer that these things exist.

Of course, whether an abductive inference is a good one depends in part on whether we have got the right observation (or whether the phenomenon in question really calls for explanation) and whether the putative explanation advanced is in fact the best one. You can decide for yourself whether the abductive argument above is in fact a good one, but we will also not be interested in abductive inferences in this book.

1.5 Arguments with several steps

Consider these two inferences:

If I left my keys at the office, Sam would have told me so.
Sam didn’t tell me that I left my keys at the office.
∴ I didn’t leave my keys at the office.

I left my keys either in the car or at the office.
I didn’t leave my keys at the office.
∴ I left my keys in the car.

Notice that the *conclusion* of the first inference is also a *premise* of the second. We can join these two inferences into a single argument with two steps. This two-step argument can be written out like this:

(1) If I left my keys at the office, Sam would have told me so. (Premise)
(2) Sam didn't tell me that I left my keys at the office. (Premise)
(3) I didn't leave my keys at the office. (From 1 and 2)
(4) I left my keys either in the car or at the office. (Premise)
(5) I left my keys in the car. (From 3 and 4)

More generally, an ARGUMENT is either a single inference, or a number of inferences combined together in sequence in the manner just described. An inference can also be called a 'one-step argument'; an argument consisting of more than one inference can be called a 'multi-step argument'.

When evaluating multi-step arguments, it is usually a good strategy to break the argument into its component steps, and to evaluate them separately.

1.6 Other logical notions

In addition to deductive validity, we will be interested in some other logical concepts.

Truth-values

As we mentioned in the previous section, a statement is a sentence (or part of a sentence) that can either be true or false. True or false are said to be the TRUTH-VALUE of a statement. So, we could have said instead that statements are sentences (or parts of sentences) that can have truth-values.

Logical truth

In considering arguments formally, we care about what would be true *if* the premises were true. Generally, we are not concerned with the actual truth-value of any particular sentences—whether they are *actually* true or false. Yet there are some sentences that must be true, just as a matter of logic.

Consider these sentences:

1. It is raining.
2. Either it is raining, or it is not.
3. It is both raining and not raining.

In order to know if sentence 1 is true, you would need to look outside or check the weather channel. Logically speaking, it might be either true or false. Sentences like this are called *contingent* sentences.

Sentence 2 is different. You do not need to look outside to know that it is true. Regardless of what the weather is like, it is either raining or not. This sentence is *logically true*; it is true merely as a matter of logic, regardless of what the world is actually like. As we shall see in chapter 3, this particular kind of logical truth is called a TAUTOLOGY. But we must be careful here, since as we will see in subsequent chapters, not all logical truths are tautologies.

You do not need to check the weather to know about sentence 3 either. It must be false, simply as a matter of logic. It might be raining here and not raining across town, it might be raining now but stop raining even as you read this, but it is impossible for it to be both raining and not raining here at this moment. The third sentence is *logically false*; it is false regardless of what the world is like. A logically false sentence is called a LOGICAL FALSEHOOD. Again, as we will see in chapter 3, this particular kind of logical falsehood is called a 'contradiction', or a 'tautological falsehood'.

To be precise, we can define a CONTINGENT SENTENCE as a sentence that is neither a logical truth nor a logical falsehood.

A sentence might *always* be true and still be contingent. For instance, if there never were a time when the universe contained fewer than seven things, then the sentence 'At least seven things exist' would always be true. Yet the sentence is contingent; its truth is not a matter of logic. It is possible to conceive of a world in which there are fewer than seven things. The important question is whether the sentence *must* be true, just on account of logic.

Logical equivalence

We can also ask about the logical relations *between* two sentences. For example:

> John went to the store after he washed the dishes.
> John washed the dishes before he went to the store.

These two sentences are both contingent, since John might not have gone to the store or washed the dishes at all. Yet they must have the same truth-value. If either of the sentences is true, then they both are; if either of the sentences is false, then they both are. When two sentences necessarily have the same truth-value, we say that they are LOGICALLY EQUIVALENT.

Consistency

Consider these two sentences:

B1 My only brother is taller than I am.
B2 My only brother is shorter than I am.

Logic alone cannot tell us which, if either, of these sentences is true. Yet we can say that *if* the first sentence (B1) is true, *then* the second sentence (B2) must be false. And if B2 is true, then B1 must be false. It cannot be the case that both of these sentences are true.

If a set of sentences could not all be true at the same time, like B1—B2, they are said to be INCONSISTENT. Otherwise, they are CONSISTENT.

We can ask about the consistency of any number of sentences. For example, consider the following list of sentences:

G1 There are at least four giraffes at the wild animal park.
G2 There are exactly seven gorillas at the wild animal park.
G3 There are not more than two martians at the wild animal park.
G4 Every giraffe at the wild animal park is a martian.

G1 and G4 together imply that there are at least four martian giraffes at the park. This conflicts with G3, which implies that there are no more than two martian giraffes there. So the set of sentences G1—G4 is inconsistent. Notice that the inconsistency has nothing at all to do with G2. G2 just happens to be part of an inconsistent set.

Sometimes, people will say that an inconsistent set of sentences 'contains a contradiction.' By this, they mean that it would be logically impossible for all of the sentences to be true at once. Notice that if the set of sentences contains *at least one* sentence that is a logical falsehood, then the set would automatically be inconsistent since it would be logically impossible for all of the sentences in that set to be true at the same time. However, even if the set of sentences does not contain a logical falsehood, this does not mean that the set is automatically consistent—such a set may or may not be consistent. Indeed, if the set contains at least two contingent sentences, then it could be inconsistent. For instance, the set containing the contingent sentences "it is raining" and "it is not raining" is inconsistent since it is logically impossible for both these sentences to be true at the same time. On the other hand, if *all* the sentences in a set are logical truths, then that set would automatically be consistent.

1.7 Validity and logical form

Consider the following three inferences:

Every even number is the sum of two prime numbers.
146 is an even number.
∴ 146 is the sum of two prime numbers.

Every frog is an amphibian.
Philip is a frog.
∴ Philip is an amphibian.

Every alkane is a hydrocarbon.
Butane is an alkane.
∴ Butane is a hydrocarbon.

These three arguments have something in common—a shared structure, or a shared form. We might put it like this:

Every A is a B.
x is an A.
∴ x is a B.

I hope you can see that every argument with this form is valid. This means that you can tell that an argument is valid by noticing that it has this form—even if you know nothing about the particular subject matter of the inference. You don't need to know anything about chemistry to know that the above argument about butane is valid.

The logical structure of an inference is not always easy to discern when the inference is written in a natural language like English (or Mandarin, or French, or whatever). For this reason, when we study logic we use 'formal languages' instead. When evaluating an inference written out in English, a logician will often 'translate' the inference into a formal language, to get a better understanding of its structure.

Summary of logical notions

- An inference is (deductively) VALID if it is impossible for the premises to be true and the conclusion false; it is INVALID otherwise.
- A LOGICAL TRUTH is a sentence that must be true, as a matter of logic.
- A TAUTOLOGY is a particular kind of logical truth.
- A LOGICAL FALSEHOOD is a sentence that must be false, as a matter of logic.
- A CONTRADICTION is a particular kind of logical falsehood.
- A CONTINGENT SENTENCE is neither a logical truth nor logical falsehood.
- Two sentences are LOGICALLY EQUIVALENT if they necessarily have the same truth-value.
- A set of sentences is CONSISTENT if it is logically possible for all the members of the set to be true at the same time; it is INCONSISTENT otherwise.

Practice exercises

Part A Which of the following are 'statements' in the logical sense?

1. England is smaller than China.
2. Greenland is south of Jerusalem.
3. Is New Jersey east of Wisconsin?
4. The atomic number of helium is 2.
5. The atomic number of helium is π.
6. I hate overcooked noodles.
7. Blech! Overcooked noodles!
8. Overcooked noodles are disgusting.
9. Take your time.
10. This is the last question.

Part B For each of the following: Is it a logical truth, a logical falsehood, or a contingent statement?

1. Caesar crossed the Rubicon.
2. Someone once crossed the Rubicon.
3. No one has ever crossed the Rubicon.
4. If Caesar crossed the Rubicon, then someone has.
5. Even though Caesar crossed the Rubicon, no one has ever crossed the Rubicon.
6. If anyone has ever crossed the Rubicon, it was Caesar.

Part C Look back at the sentences G1—G4 on p. 14, and consider each of the following sets of sentences. Which are consistent? Which are inconsistent?

1. G2, G3, and G4
2. G1, G3, and G4
3. G1, G2, and G4
4. G1, G2, and G3

Part D Which of the following is possible? If it is possible, give an example. If it is not possible, explain why.

1. A valid inference that has one false premise and one true premise
2. A valid inference that has a false conclusion
3. A valid inference, the conclusion of which is a logical falshood
4. An invalid inference, the conclusion of which is a tautology
5. A logical truth that is contingent
6. Two logically equivalent sentences, both of which are logical truths
7. Two logically equivalent sentences, one of which is a logical truth and one of which is contingent
8. Two logically equivalent sentences that together are an inconsistent set
9. A consistent set of sentences that contains a contradiction
10. An inconsistent set of sentences that contains a tautology

Part E Look at the following passage. How many inferences can you find? Which of these inferences are valid?

> Ashni told me that she only sees Ben once or twice a year, so they must live far apart. They used to hang out so often. It's sad, really. Anyway, Ashni and Ben don't both live in New York. So since Ashni lives in New York, Ben doesn't.

Part I

Truth-functional logic

Chapter 2

Atomic sentences and the Boolean connectives

This chapter and the next together introduce a logical language called 'TFL'. It is a version of *truth-functional logic* which is expressed in a fragment of first-order logic (FOL) and focuses on sentences whose truth or falsity solely depends on the truth or falsity of their most basic parts.

2.1 Atomic sentences

We begin by considering how to construct *atomic sentences* in TFL, which correspond to some of the most basic statements we can make in natural languages like English. Consider the following basic English sentence:

1. Ashni is tall.

This sentence is composed of two parts: a name ('Ashni') and a verb phrase or predicate 'is tall'. We can say that the verb phrase 'is tall' is predicating the property of tallness to Ashni.

In TFL, atomic sentences are also composed of two parts: First, we have INDIVIDUAL CONSTANTS , which are simple terms and are the TFL analogue to names in English. Second, we have PREDICATES, which are the TFL analogue to verb phrases or predicates in English, and are used to refer to properties of objects or relations between them. In TFL, we usually use uppercase letters to represent predicates, and lowercase letters to represent names of objects. (There are some exceptions to this rule, as we will see below.) This convention helps distinguish symbols or letters that express predicates from those that name objects. For example, we might decide to use the uppercase letter '*T*' to stand for the predicate 'is tall' and the lowercase letter '*a*' to stand for the name 'Ashni', in which case we could translate 1 above into TFL in the following way:

2. *Ta*

We used '*T*' and '*a*' as letters to represent the predicate 'is tall' and the name 'Ashni' respectively, because they are the first letters of the English predicate ('is **t**all') and name ('**A**shni'). This often helps with remembering the meanings of the symbols (i.e., the letters) used in TFL. But we could have used different letters, as long as we take care to explain what these symbols mean. For example, we could have offered the following SYMBOLIZATION KEY:

***Bx*:** *x* is tall
***m*:** Ashni

We could have then translated sentence 1 above in the following way:

3. *Bm*

Admittedly, it is a bit odd to translate 1 into 3 rather than into 2. But as long as we have an adequate symbolization key—i.e., a list of symbols used in the translation along with their meanings—there is no risk of confusion. Again, consider the following sentence:

4. Ben lives in Japan.

Now consider the following symbolization key:

***Lx*:** *x* lives in Japan
***b*:** Ben

We are now in a position to translate 4 into TFL:

5. *Lb*

Not all basic English sentences lend themselves so easily to this method of translation. Consider the following sentence:

6. It is raining outside.

Like 1 and 4, 6 contains a predicate: 'is raining'. Unlike 1 and 4, however, it does not contain a name. So how are we supposed to translate 6 into TFL? Here, we are going to allow ourselves a certain amount of leeway. We can treat 'outside' as the name of a location. Then, with the following symbolization key:

$\boldsymbol{Rx}$**:** x is raining
$\boldsymbol{o}$**:** outside

we can translate 6 into TFL in the following way:

7. Ro

Technically, 7 says 'Outside is raining', which isn't entirely grammatical, but we will allow it to stand for an adequate translation of 6.

There are only twenty-six letters in the alphabet, but we never need run out of letters for individual constants and predicates, because when necessary we can use the same letter to symbolize different constants and predicates by adding a subscript, a small number written after the letter. We could have a symbolization key that looks like this:

$\boldsymbol{P_1x}$**:** x is tall
$\boldsymbol{P_2x}$**:** x lives in Japan
$\boldsymbol{P_3x}$**:** x is raining
$\boldsymbol{a_1}$**:** Ashni
$\boldsymbol{a_2}$**:** Ben
$\boldsymbol{a_3}$**:** outside

Before we move on to section 2.2 and discuss the ways in which we can combine atomic sentences in TFL to build complex sentences, a few more points are in order.

First, one way in which TFL differs from natural languages like English has to do with the way individual constants are used in TFL to name objects. In a language like English, it is perfectly acceptable for more than one person to share the same name. In TFL, however, we insist that no individual constant refers to more than one object. We also insist that individual constants actually refer to existing objects. It is perfectly fine, however, if some object has more than one name (i.e., if two individual constants refer to the same object), or if some objects have no name.

Second, we said earlier that we can use lowercase letters as individual constants. But we should qualify this here. We will exclude among the lowercase letters that we can use as individual constants the last three letters of the alphabet: 'x', 'y', 'z'. We will reserve these for chapter 7 when we introduce quantification and use them as *variables*.

Third, consider again the predicate symbol used to express 'lives in Japan' in sentence 4. In order to make an atomic sentence out of 'L', all we did was proceed it with one individual constant 'b'. That is because the ARITY of the predicate 'L' is one. The arity of a predicate is simply the number of argument places that need to be filled by terms in order for the resulting expression to be an atomic sentence. So, 'L' is a unary predicate: it has only one argument place (i.e., one placeholder) for terms. This is something we determined implicitly when we settled on our symbolization key used to translate sentence 4. But we could have made a different choice. Notice that the sentence 'Ben lives in Japan' expresses a relation between two things, namely, Ben and Japan. So, in translating sentence 4, we could have developed a symbolization key which treats 'lives in' as a binary relation, and added an individual constant to name Japan:

Lxy: x lives in y
b: Ben
j: Japan

The translation would be:

8. Lbj

In this course, we will focus on *unary* predicates and *binary* relations only, but TFL can also deal with ternary (arity of 3) and quaternary (arity of 4) relations (or any n-ary relation, really). Consider sentence:

9. Louise is sitting between Farouk and Magda.

Here, the predicate 'is sitting between' relates three objects, namely Louise, Farouk, and Magda. Given a proper symbolization key, we might translate the sentence into TFL thusly:

10. $Blfm$

Or consider the sentence:

11. Talia is in front of June, Desmond, and Ellie.

Here, we can treat 'is in front of' as a quaternary relation between Talia, June, Desmond, and Ellie. Again, given a proper symbolization key, we might translate the sentence into TFL thusly:

12. $Ftjde$

An ATOMIC SENTENCE, then, is any n-ary predicate followed by n terms.

Fourth, you will notice that the predicate symbols (i.e., the uppercase letters) used in sentences 2, 3, 5, 7, 8, 10 and 12 all precede their constants. This way of writing out sentences in TFL uses what is called 'prefix notation'. For the most part, we will stick with this convention and use prefix notation. But we will make an exception with one binary relation: the relation of identity. In fact, we will make three exceptions with regards to this relation. First, we will use '=' rather than an uppercase letter to express the identity relation in TFL. The reason for this should be obvious: we are already familiar with this symbol and its meaning. Second, when we want to translate identity statements—statements like 'Samuel Clemens is Mark Twain' or '2 is identical to 1'—we shall use what is called 'infix notation' instead of prefix notion. Again, the reason for this obvious: we are already familiar with expressions like '$a = b$' in a way that makes '$= ab$' look a bit odd. Third, we will exclusively use '=' to express the identity relation. So, if we want to make an identity claim in TFL, we will use '='. Given a proper symbolization key, the sentences above could be translated respectively into TFL thusly:

13. $c = t$
14. $2 = 1$

Sentence 14 leads to the final point. Earlier we said that we will use lowercase letters (except for 'x', 'y', and 'z') as individual constants. The only exception will be when we want to make certain kinds of mathematical statements, like statements in arithmetic. There, we will allow ourselves to use the Hindu-Arabic numerals (i.e., 1, 2, 3, and so on) as individual constants to refer to the natural numbers.

2.2 Connectives

Logical connectives are used to build complex sentences from atomic components. There are five logical connectives in TFL. This table summarizes them.

symbol	what it is called	what it means
$\neg$	negation	'It is not the case that...'
$\wedge$	conjunction	'Both... and ...'
$\vee$	disjunction	'Either... or ...'
$\rightarrow$	conditional	'If ... then ...'
$\leftrightarrow$	biconditional	'... if and only if ...'

In this chapter, we will discuss negation, conjunction, and disjunction. These three connectives are called the 'Boolean connectives', because the first person to give them a systematic study was 19th century British mathematician and logician George Boole. We will discuss conditionals and biconditionals in a later chapter.

Negation

Consider how we might symbolize these sentences:

1. Mary is in Barcelona.
2. Mary is not in Barcelona.
3. Mary is somewhere besides Barcelona.

In order to symbolize sentence 1, we will need one predicate symbol (i.e., an uppercase letter) and one individual constant (i.e., a lowercase letter). We can provide the following symbolization key:

Bx: x is in Barcelona.
m: Mary

Note that we are here giving B and m different interpretations than we did in the previous section. The symbolization key only specifies what symbols mean *in a specific context*. It is vital that we continue to use this meaning of B and m so long as we are talking about Mary and Barcelona. Later, when we are symbolizing different sentences, we can write a new symbolization key and use B and m to mean something else.

Now, sentence 1 is simply Bm.

Since sentence 2 is obviously related to the sentence 1, we do not want to introduce different predicate and constant symbols. To put it partly in English, the sentence means 'Not *Bm*.' In order to symbolize this, we need a symbol for logical NEGATION. We will use '¬'. Now we can translate 'Not *Bm*' to $\neg Bm$.

Sentence 3 is about whether or not Mary is in Barcelona, but it does not contain the word 'not.' Nevertheless, it is obviously logically equivalent to sentence 2. They both mean: It is not the case that Mary is in Barcelona. As such, we can translate both sentence 2 and sentence 3 as $\neg Bm$.

> A sentence can be symbolized as $\neg\mathcal{P}$ if it can be paraphrased in English as 'It is not the case that $\mathcal{P}$.' '$\mathcal{P}$' is called a sentential letter. Here and elsewhere, it stands for any sentence, atomic or non-atomic.

Consider these further examples:

4. The widget can be replaced if it breaks.
5. The widget is irreplaceable.
6. The widget is not irreplaceable.

Sentence 4 can be paraphrased as 'The widget is replaceable'. Given the following symbolization key:

Rx: x is replaceable
w: the widget

sentence 4 can be translated into TFL as Rw.

What about sentence 5? Saying the widget is irreplaceable means that it is not the case that the widget is replaceable. So even though sentence 5 is not negative in English, we symoblize it using negation as $\neg Rw$.

Sentence 6 can be paraphrased as 'It is not the case that the widget is irreplaceable.' Using negation twice, we translate this as $\neg\neg Rw$. The two negations in a row each work as negations, so the sentence means 'It is not the case that… it is not the case that… Rw.' If you think about the sentence in English, it is logically equivalent to sentence 4. So when we define logical equivalence in TFL, we will make sure that Rw and $\neg\neg Rw$ are logically equivalent.

More examples:

7. Elliott is happy.
8. Elliott is unhappy.

If we let Hx mean 'x is happy' and e name Elliot, then we can symbolize sentence 7 as He.

However, it would be a mistake to symbolize sentence 8 as $\neg He$. If Elliott is unhappy, then he is not happy—but sentence 8 does not mean the same thing as 'It is not the case that Elliott is happy.' It could be that he is not happy but that he is not unhappy either. Perhaps he is somewhere between the two. In order to allow for the possibility that he is indifferent, we would need a new sentence letter to symbolize sentence 8.

For any sentence $\mathcal{P}$: If $\mathcal{P}$ is true, then $\neg\mathcal{P}$ is false. If $\neg\mathcal{P}$ is true, then $\mathcal{P}$ is false. Using 'T' for true and 'F' for false, we can summarize this in a *characteristic truth table* for negation:

$\mathcal{P}$	$\neg\mathcal{P}$
T	F
F	T

We will discuss truth tables at greater length in the next chapter.

Conjunction

Consider these sentences:

9. Adam is athletic.
10. Barbara is athletic.
11. Adam is athletic, and Barbara is also athletic.

Let us use the following symbolization key:

***Ax*:** *x* is athletic.
***d*:** Adam
***b*:** Barbara

Note that '*d*' is used to name Adam instead of '*a*' in order to avoid confusing it with the predicate symbol for 'is athletic', and because '*d*' is the second letter in Adam's name. Sentence 9 can be symbolized as Ad.

Sentence 10 can be symbolized as Ab.

Sentence 11 can be paraphrased as 'Ad and Ab.' In order to fully symbolize this sentence, we need another symbol. We will use '$\wedge$'. We translate 'Ad and Ab' as $(Ad \wedge Ab)$. The logical connective '$\wedge$' is called CONJUNCTION, and Ad and Ab are each called CONJUNCTS.

Notice that we make no attempt to symbolize 'also' in sentence 11. Words like 'both' and 'also' function to draw our attention to the fact that two things are being conjoined. They are not doing any further logical work, so we do not need to represent them in TFL.

Some more examples:

12. Barbara is athletic and energetic.
13. Barbara and Adam are both athletic.
14. Although Barbara is energetic, she is not athletic.
15. Barbara is athletic, but Adam is more athletic than she is.

Sentence 12 is obviously a conjunction. The sentence says two things about Barbara, so in English it is permissible to refer to Barbara only once. It might be tempting to try this when translating the argument: Since Ab means 'Barbara is athletic', one might paraphrase the sentences as 'Ab and energetic.' This would be a mistake. Once we translate part of a sentence as Ab, any further structure is lost. Ab is an atomic sentence; it is nothing more than true or false. Conversely, 'energetic' is not a sentence; on its own it is neither true nor false. We should instead paraphrase the sentence as 'Ab and Barbara is energetic.' Now we need to add a sentence letter to the symbolization key. Let E mean 'x is energetic.' Now the sentence can be translated as $(Ab \land Eb)$.

A sentence can be symbolized as $(\mathcal{P} \land \mathcal{Q})$ if it can be paraphrased in English as 'Both $\mathcal{P}$, and $\mathcal{Q}$'. Each of the conjuncts must be a sentence. As with '$\mathcal{P}$', '$\mathcal{Q}$' is a sentential letter. It stands for any sentence whatsoever.

Sentence 13 says one thing about two different subjects. It says of both Barbara and Adam that they are athletic, and in English we use the word 'athletic' only once. In translating to TFL, it is important to realize that the sentence can be paraphrased as, 'Barbara is athletic, and Adam is athletic.' This translates as $(Ab \land Ad)$.

Sentence 14 is a bit more complicated. The word 'although' sets up a contrast between the first part of the sentence and the second part. Nevertheless, the sentence says both that Barbara is energetic and that she is not athletic. In order to make each of the conjuncts an atomic sentence, we need to replace 'she' with 'Barbara.'

So we can paraphrase sentence 14 as, '*Both* Barbara is energetic, *and* Barbara is not athletic.' The second conjunct contains a negation, so we paraphrase further: '*Both* Barbara is energetic *and it is not the case that* Barbara is athletic.' This translates as $(Eb \land \neg Ab)$.

Sentence 15 contains a similar contrastive structure. It is irrelevant for the purpose of translating to TFL, so we can paraphrase the sentence as '*Both* Barbara is athletic, *and* Adam is more athletic than Barbara.' (Notice that we once again replace the pronoun 'she' with her name.) How should we translate the second conjunct? We don't have a predicate symbol in our symbolization key expressing the relation that one thing is more athletic than another. So, we need to add one:

Mxy: x is more athletic than y

Note that M is a binary relation, and so allows us to translate sentence 15 into $(Ab \wedge Mdb)$.

> Sentences that can be paraphrased '$\mathcal{P}$, but $\mathcal{Q}$' or 'Although $\mathcal{P}$, $\mathcal{Q}$' are best symbolized using conjunction: $(\mathcal{P} \wedge \mathcal{Q})$.

Two things are worth keeping in mind here. First, we are using expressions like '$\mathcal{P}$' and '$\mathcal{Q}$' to stand for atomic sentences. Similarly, we used '$\mathcal{P}$' to stand for an atomic sentence when we constructed the characteristic truth table for negation. These expressions are not really atomic sentences. Rather, they stand for *any* atomic sentence with *any* n-ary predicate, including atomic sentences with binary predicates like the one introduced above.

Second, the sentences Ab, Ad, Eb, and Mdb are atomic sentences. Considered as symbols of TFL, they have no meaning beyond being true or false. We have used them to symbolize different English language sentences that are all about people being athletic, but this similarity is completely lost when we translate to TFL. No formal language can capture all the structure of the English language, but as long as this structure is not important to the argument there is nothing lost by leaving it out.

For any sentences $\mathcal{P}$ and $\mathcal{Q}$, $(\mathcal{P} \wedge \mathcal{Q})$ is true if and only if both $\mathcal{P}$ and $\mathcal{Q}$ are true. We can summarize this in the characteristic truth table for conjunction:

$\mathcal{P}$	$\mathcal{Q}$	$(\mathcal{P} \wedge \mathcal{Q})$
T	T	T
T	F	F
F	T	F
F	F	F

Conjunction is *symmetrical* because we can swap the conjuncts without changing the truth-value of the sentence. Regardless of what $\mathcal{P}$ and $\mathcal{Q}$ are, $(\mathcal{P} \wedge \mathcal{Q})$ is logically equivalent to $(\mathcal{Q} \wedge \mathcal{P})$.

Disjunction

Consider these sentences:

16. Either Denison will play golf with me, or he will watch movies.
17. Either Denison or Ellery will play golf with me.

For these sentences we can use this symbolization key:

***Gxy*:** *x* will play golf with *y*
***Mx*:** *x* will watch movies
***d*:** Denison
***e*:** Ellery
***a*:** me

Sentence 16 can be paraphrased as 'Either *Gda* or *Md*.' To fully symbolize this, we introduce a new symbol '∨'. The sentence becomes (*Gda* ∨ *Md*). The '∨' connective is called DISJUNCTION, and *Gda* and *Md* are called DISJUNCTS.

Sentence 17 is only slightly more complicated. There are two subjects, but the English sentence only gives the verb once. In translating, we can paraphrase it as: 'Either Denison will play golf with me, or Ellery will play golf with me.' Now it obviously translates as (*Gda* ∨ *Gea*).

A sentence can be symbolized as ($\mathcal{P} \vee \mathcal{Q}$) if it can be paraphrased in English as 'Either $\mathcal{P}$, or $\mathcal{Q}$.' Each of the disjuncts must be a sentence.

Sometimes in English, the word 'or' excludes the possibility that both disjuncts are true. This is called an EXCLUSIVE OR. An *exclusive or* is clearly intended when it says, on a restaurant menu, 'Entrees come with either soup or salad.' You may have soup; you may have salad; but, if you want *both* soup *and* salad, then you have to pay extra.

At other times, the word 'or' allows for the possibility that both disjuncts might be true. This is probably the case with sentence 17, above. I might play with Denison, with Ellery, or with both Denison and Ellery. Sentence 17 merely says that I will play with *at least* one of them. This is called an INCLUSIVE OR.

The symbol '∨' represents an *inclusive or*. So (*Gda* ∨ *Gea*) is true if *Gda* is true, if *Gea* is true, or if both *Gda* and *Gea* are true. It is false only if both *Gda* and *Gea* are false. We can summarize this with the characteristic truth table for disjunction:

$\mathcal{P}$	$\mathcal{Q}$	($\mathcal{P} \vee \mathcal{Q}$)
T	T	T
T	F	T
F	T	T
F	F	F

Like conjunction, disjunction is symmetrical. $(\mathcal{P} \lor \mathcal{Q})$ is logically equivalent to $(\mathcal{Q} \lor \mathcal{P})$.

These sentences are somewhat more complicated:

18. Either you will not have soup, or you will not have salad.
19. You will have neither soup nor salad.
20. You get either soup or salad, but not both.

Let us use the following symbolization key:

S_1x: x will have soup
S_2x: x will have salad
u: you

Sentence 18 can be paraphrased in this way: 'Either *it is not the case that* you get soup, or *it is not the case that* you get salad'. Translating this requires both disjunction and negation. It becomes $(\neg S_1u \lor \neg S_2u)$.

Sentence 19 also requires negation. It can be paraphrased as, '*It is not the case that* either that you get soup or that you get salad'. We need some way of indicating that the negation does not just negate the right or left disjunct, but rather negates the entire disjunction. In order to do this, we put parentheses around the disjunction: 'It is not the case that $(S_1u \lor S_2u)$'. This becomes simply $\neg(S_1u \lor S_2u)$.

Sentence 20 is an *exclusive or*. We can break the sentence into two parts. The first part says that you get one or the other. We translate this as $(S_1u \lor S_2u)$. The second part says that you do not get both. We can paraphrase this as, 'It is not the case both that you get soup and that you get salad.' Using both negation and conjunction, we translate this as $\neg(S_1u \land S_2u)$. Now we just need to put the two parts together. As we saw above, 'but' can usually be translated as a conjunction. Sentence 20 can thus be translated as $\big((S_1u \lor S_2u) \land \neg(S_1u \land S_2u)\big)$.

2.3 Parentheses matter

Consider these two sentences:

21. Mijung isn't both at home *and* at the movie theatre.
22. Mijung isn't at home but she *is* at the movie theatre.

We let H mean 'x is at home', M mean 'x is at the movie theatre' and i name Mijung.

Sentence 21 asserts that Mijung isn't simultaneously at home *and* at the movie theatre—which is presumably true, unless Mijung lives at the theatre. This sentence can be translated $\neg(Hi \wedge Mi)$. Sentence 22 asserts first that Mijung is not at home (which can be translated $\neg Hi$) and second that Mijung is at the movie theatre (which is just Mi). So sentence 22 as a whole can be translated $(\neg Hi \wedge Mi)$.

Now suppose that Mijung is at home (and not at the movie theatre). In this case, sentence 21 true but sentence 22 is false. This means that $\neg(Hi \wedge Mi)$ and $(\neg Hi \wedge Mi)$ can have opposite truth values, even though they look very similar. This example shows that the positions of the parentheses in a statement can make a big difference!

Summary of logical notions

- An INDIVIDUAL CONSTANT is a *singular term* that uniquely refers to an existing object.
- A PREDICATE is a symbol that refers to a property of an object or a relation between objects.
- The ARITY of a predicate is the number of argument places that need to be filled by terms in order for the resulting expression to be an atomic sentence.
- An ATOMIC SENTENCE in TFL is any n-ary predicate followed by n terms.
- A SYMBOLIZATION KEY is a list of letters or symbols and their meanings used in translating sentences from a natural language like English into a logical language like TFL.
- NEGATION is a unary Boolean connective that operates on a sentence, atomic or complex, forming a complex sentence.
- CONJUNCTION is a binary Boolean connective that operates on two sentences, atomic or complex, forming a complex sentence. The two parts of a conjunction are called CONJUNCTS.
- DISJUNCTION is a binary Boolean connective that operates on two sentences, atomic or complex, forming a complex sentence. The two parts of a disjunction are called DISJUNCTS.
- A sentence that uses an EXCLUSIVE OR is true if and only if either disjuncts are true but not both.
- A sentence that uses an INCLUSIVE OR is false if and only if both disjuncts are false.

Practice exercises

Part A Using the symbolization key given, translate each English-language sentence into TFL.

Mx: x is a man in a suit
Cx: x is a chimpanzee
Gx: x is a gorilla
b: Bob
k: Koko
f: Flo

1. Bob is a man in a suit.
2. Bob is a man in a suit or he is not.
3. Koko is either a gorilla or a chimpanzee.
4. Bob is neither a gorilla nor a chimpanzee.
5. Flo is neither a gorilla nor a man in a suit, and nor a chimpanzee.
6. Flo is either a gorilla or a chimpanzee, not a man in a suit.

Part B Using the symbolization key given, translate each English-language sentence into TFL. The translations build off each other, such that something established in an earlier translation is sometimes used in subsequent sentences.

M_1x: x was murdered
M_2xy: x murdered y
Lx: x is lying
Fx: x was a frying pan
a: Mister Ace
b: the butler
c: the cook
d: the Duchess
e: Mister Edge
w: the murder weapon

1. Either Mister Ace or Mister Edge was murdered.
2. Mister Ace and Mister Edge weren't both murdered.
3. Either the cook did it, or the butler did it.
4. Either the butler did it, or the Duchess is lying.
5. Mister Edge was murdered and the cook did it.
6. Either the murder weapon was a frying pan or the Duchess isn't lying.
7. Either the Duchess is lying, or the culprit is either the cook or the butler.
8. Either the Duchess is lying, or both Mister Edge and Mister Ace were murdered.
9. Neither the butler nor the cook did it.

10. Although the murder weapon was a frying pan, either both Mister Edge and Mister Ace were murdered, or the Duchess is lying.
11. Of course the Duchess is lying!

Part C Using the symbolization key given, translate each English-language sentence into TFL.

***Ex*:** x is an electrician.
***Fx*:** x is a firefighter.
***Sx*:** x is satisfied with their career.
***a*:** Ava
***h*:** Harrison

1. Ava and Harrison are both electricians.
2. Ava is a firefighter satisfied with her career.
3. Ava is either a firefighter or an electrician.
4. Harrison is an unsatisfied electrician.
5. Neither Ava nor Harrison is an electrician.
6. Both Ava and Harrison are electricians, but neither of them find it satisfying.
7. Harrison is either an unsatisfied firefighter or a satisfied electrician.
8. Harrison and Ava are both firefighters who are satisfied with their careers.
9. Either Harrison and Ava are both firefighters or neither of them is a firefighter.

Part D In the chapter, we symbolized an *exclusive or* using $\vee$, $\wedge$, and $\neg$. How could you translate an *exclusive or* using only two connectives? Is there any way to translate an *exclusive or* using only one connective?

Chapter 3

Truth tables

This chapter introduces a way of evaluating sentences and arguments of TFL. Although it can be laborious, the truth table method is a purely mechanical procedure that requires no intuition or special insight.

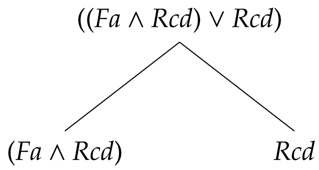
3.1 Decomposing a statement

As we've discussed, in TFL we make bigger statements from smaller ones. The smaller statements from which a larger one is composed are called "substatements". When trying to understand the structure of a statement, it is often helpful to draw a tree, showing the manner in which the substatements are combined.

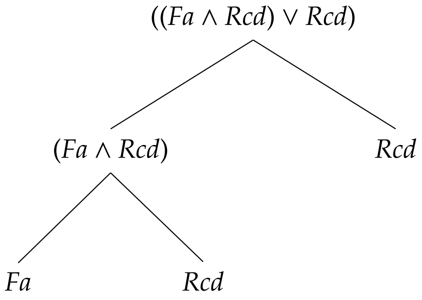
Consider for example this statement:

$$((Fa \wedge Rcd) \vee Rcd)$$

This statement is a disjunction. We can split it into its two disjuncts:

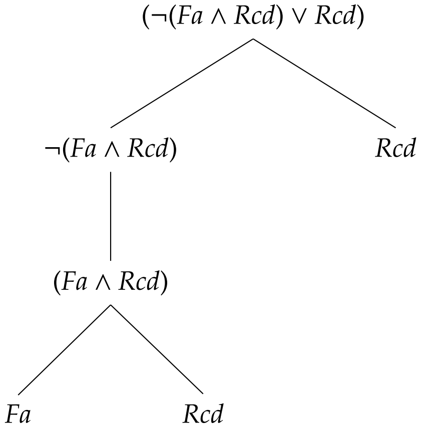


Now the left-hand disjunct can be further decomposed:



Now our tree is complete. We have completely decomposed our statement into its component parts. The tree shows all the substatements.

Here is a slightly more complex example, involving negation:



When you're drawing these trees, it's very important to remember that a statement always has the same number of left brackets and right brackets!

3.2 Truth-functional connectives

Any non-atomic sentence of TFL is composed of atomic sentences with truth-functional connectives. A connective is truth-functional when the truth value of the compound sentence depends only on the truth value of the atomic sentences that comprise it. For example, in order to know the truth value of $(a = b \vee a = c)$, we only need to know the truth value of $a = b$ and the truth value of $a = c$.

In this chapter, we will make use of the fact that all of the logical operators in TFL are truth-functional—it makes it possible to construct truth tables to determine the logical features of sentences. You should realize, however, that this is not possible for all languages. In English, it is possible to form a new sentence from any simpler sentence $\mathcal{P}$ by saying 'It is possible that $\mathcal{P}$.' The truth value of this new sentence does not depend directly on the truth value of $\mathcal{P}$. Even if $\mathcal{P}$ is false, perhaps in some sense $\mathcal{P}$ *could* have been true—then the new sentence would be true. Some formal languages, called *modal logics*, have an operator for possibility. In a modal logic, we could translate 'It is possible that $\mathcal{P}$' as $\Diamond\mathcal{P}$. However, the ability to translate sentences like these come at a cost: The $\Diamond$ operator is not truth-functional, and so modal logics are not amenable to truth tables.

3.3 Complete truth tables

The truth value of sentences which contain only one connective are given by the characteristic truth table for that connective. In the previous chapter, we wrote the characteristic truth tables with 'T' for true and 'F' for false. It is important to note, however, that this is not about truth in any deep or cosmic sense. Poets and philosophers can argue at length about the nature and significance *truth*, but the truth functions in TFL are just rules which transform input values into output values.

Here are the truth tables for the connectives we've studied so far, written in terms of Ts and Fs.

$\mathcal{P}$	$\neg\mathcal{P}$
T	F
F	T

$\mathcal{P}$	$\mathcal{Q}$	$(\mathcal{P}\wedge\mathcal{Q})$	$(\mathcal{P}\vee\mathcal{Q})$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Table 3.1: The characteristic truth tables for the connectives of TFL.

The characteristic truth table for conjunction, for example, gives the truth conditions for any sentence of the form $(\mathcal{P}\wedge\mathcal{Q})$. Even if the conjuncts $\mathcal{P}$ and $\mathcal{Q}$ are long, complicated sentences, the conjunction is true if and only if both $\mathcal{P}$ and $\mathcal{Q}$ are true.

Now let's draw a truth table for a more complicated statement, $\big((Fa\wedge Rcd)\vee Rcd\big)$. To do this, we need one row for each possible assignment of truth values to the atomic statements. Since we have two atomic sentences, and there are two truth values, we need four rows: (1) Both statements are true, (2) Fa is true and Rcd is false, (3) Fa is false and Rcd is true, (4) Both statements are false. We need a column for the statement we're interested in, and a column for every substatement of it. It's helpful to put the smaller statements on the left, and the bigger statements on the right. So we have:

Fa	Rcd	$(Fa \wedge Rcd)$	$\big((Fa \wedge Rcd) \vee Rcd\big)$
T	T		
T	F		
F	T		
F	F		

Now we can fill in the remaining columns. $(Fa \wedge Rcd)$ is a conjunction of Fa and Rcd. Remembering that a conjunction is true when both conjuncts are true, we can write "T" in the top row; remembering that a conjunction is false when either conjunct is false, we can write "F" in all the remaining rows:

Fa	Rcd	$(Fa \wedge Rcd)$	$\big((Fa \wedge Rcd) \vee Rcd\big)$
T	T	T	
T	F	F	
F	T	F	
F	F	F	

Now for the last column. Remembering that a disjunction is true when one of the disjuncts is true, we can write a "T" in the first row and the third row; remembering that a disjunction is false when both disjuncts are false, we can write a "F" in the second row and the fourth row:

Fa	Rcd	$(Fa \wedge Rcd)$	$\big((Fa \wedge Rcd) \vee Rcd\big)$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	F

Now our truth table is complete. One thing we notice is that the column for $((Fa \wedge Rcd) \vee Rcd)$ and the column for Rcd have the same entries! This means that these two statements are logically equivalent—more on this later.

Now let's work through a slightly more complex example, $(\neg(Fa \wedge Rcd) \vee Rcd)$. As before, we need a column for the statement we're interested in, and a column for every substatement; we again need four rows:

Fa	Rcd	$(Fa \wedge Rcd)$	$\neg\,(Fa \wedge Rcd)$	$\big(\neg(Fa \wedge Rcd) \vee Rcd\big)$
T	T			
T	F			
F	T			
F	F			

And now we can fill in the three empty columns. As before, we start at the left and move right:

Fa	Rcd	$(Fa \wedge Rcd)$	$\neg\,(Fa \wedge Rcd)$	$\big(\neg(Fa \wedge Rcd) \vee Rcd\big)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

One thing we notice is that out statement has a 'T' in every row. This means that it is a tautology—more on this later.

A COMPLETE TRUTH TABLE has a row for all the possible combinations of T and F for all of the atomic sentences. The size of the complete truth table depends on the number of different atomic sentences in the table. Atomic sentences require only two rows, as in the characteristic truth table for negation. This is true even if the same letter is repeated many times, as in the sentence $\big(Pa \wedge \neg(Pa \wedge Pa)\big)$. The complete truth table requires only two lines because there are only two possibilities: *Pa* can be true or it can be false. A single atomic sentence can never be marked both T and F on the same row. The truth table for this sentence looks like this:

Pa	$(Pa \wedge Pa)$	$\neg\,(Pa \wedge Pa)$	$\big(Pa \wedge \neg(Pa \wedge Pa)\big)$
T	T	F	F
F	F	T	F

Looking at the column underneath the main connective, we see that the sentence is false on both rows of the table; i.e., it is false regardless of whether *Pa* is true or false. This means that the statement is a contradiction—more on this later.

A sentence that contains two atomic sentences requires four rows for a complete truth table, as we have seen.

A sentence that contains three atomic sentences requires eight rows. For example:

Ma	Nb	Pc	$(Nb \vee Pc)$	$\big(Ma \wedge (Nb \vee Pc)\big)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

A complete truth table for a sentence that contains four different atomic sentences requires 16 rows. Five atomic sentences, 32 rows. Six atomic sentences, 64 rows. And so on. To be perfectly general: If a complete truth table has n different atomic sentences, then it must have 2^n rows.

In order to fill in the columns of a complete truth table, begin with the right-most atomic sentence and alternate Ts and Fs. In the next column to the left, write two Ts, write two Fs, and repeat. For the third atomic sentence, write four Ts followed by four Fs. This yields an eight row truth table like the one above. For a 16 row truth table, the next column of atomic sentences should have eight Ts followed by eight Fs. For a 32 row table, the next column would have 16 Ts followed by 16 Fs. And so on.

3.4 Using truth tables

Tautologies and contradictions

Recall from chapter 1 that a logical truth is a sentence that must be true as a matter of logic. There, we said that a tautology is a particular kind of logical truth, and that not all logical truths are tautologies. Similarly, we said that a contradiction is a particular kind of logical falsehood, but that not all logical falsehoods are contradictions. With truth tables, we are now in a position to see why. A TAUTOLOGY is simply a sentence whose truth table receives a T in every row of the column under its main connective. Similarly, a CONTRADICTION is any sentence whose truth table receives an F in every row of the column under its main connective. A sentence that receives a mixture of Ts and Fs is called a TAUTOLOGICAL CONTINGENCY

From the truth tables in the previous section, we know that $(\neg(Fa \wedge Rcd) \vee Rcd)$ is a tautology, and that $(Pa \wedge \neg(Pa \wedge Pa))$ is a contradiction.

But now consider the following sentence:

$$(a = a \land b = b)$$

This sentence is a logical truth. It must be true as a matter of logic. (Remember that we stipulated in chapter 1 that individual constants uniquely refer to existing objects.) But it is not a tautology. A completed truth table for this sentence will reveal this:

$a = a$	$b = b$	$(a = a \land b = b)$
T	T	T
T	F	F
F	T	F
F	F	F

We see quite clearly that $(a = a \land b = b)$ is not a tautology: its completed truth table does not receive a T in every row of the column under the main connective. This shows that not every logical truth is a tautology. So, constructing completed truth tables to check for logical truths has its limit: it cannot identify all logical truths.

Similarly, consider the sentence:

$$(\neg a = a \land \neg b = b)$$

We will introduce a symbol "≠" to express the negation of an identity statement, but note that this is not a new logical symbol. It is simply a shorthand for applying the negation symbol "¬" to an identity statement of the form $n = n$. So, we can express the above sentence in the following way:

$$(a \neq a \land b \neq b)$$

This sentence is a logical falsehood. It is false as a matter of logic. But it is not a contradiction, as its completed truth table will show:

$a = a$	$b = b$	$a \neq a$	$b \neq b$	$(a \neq a \land b \neq b)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

We see here that this sentence's completed truth table does not receive an F in every row of the column under its main connective. This shows that not every logical falsehood is a contradiction. And here we have another example of the limit of completed truth tables: they cannot identify all logical falsehoods.

Why is it that completed truth tables are limited in this way? The culprit, as you might have guessed, is the identity symbol. We know what it means: 'is identical to' and this meaning is fixed in PL. But its meaning is rendered completely opaque when we put identity statements in truth tables. That is because truth tables are only sensitive to the meanings of the truth functional connectives. In terms of truth tables, there is no difference between a statement like $a = a$ and any other statement that expresses a binary relation between two objects—e.g., Raa—and which is not a logical truth. *We* know that any row that assigns an F to $a = a$ or to $b = b$, or a T to $a \neq a$ or $b \neq b$ does not correspond to a possible circumstance, but our truth table method does not 'know' this.

Tautological equivalence

Two sentences are logically equivalent if they have the same truth value as a matter logic. With truth tables, we can make this notion more precise in TFL. You can show in TFL that two statements are TAUTOLOGICALLY EQUIVALENT by drawing a single truth table containing both statements, what we will call a 'joint truth table'. If the two statements have the same entries in every row, then they are tautologically equivalent.

Consider the sentences $\neg(Rab \lor Tc)$ and $(\neg Rab \land \neg Tc)$. Are they logically equivalent? To find out, we construct a truth table.

Rab	Tc	$(Rab \lor Tc)$	$\neg\ (Rab \lor Tc)$	$\neg\ Rab$	$\neg\ Tc$	$(\neg Rab \land \neg Tc)$
T	T	T	**F**	F	F	**F**
T	F	T	**F**	F	T	**F**
F	T	T	**F**	T	F	**F**
F	F	F	**T**	T	T	**T**

Our two statements have the same entry in every row, so they are tautologically equivalent. Two statements (or any number of statements) are *contradictory* if there is no row in their joint truth table where each statement receives a T.

In the same way that truth tables cannot pick out all logical truths and logical falsehoods, they also cannot pick out all logical equivalences. Consider the pair of sentences $a = a$ and $b = b$. These sentences are both logically true. So, they share the same truth value as a matter of logic, and are thus logically equivalent. Yet, they are not tautologically equivalent, as a joint truth table will show. (We leave it to the reader to show this using a joint truth table.) We will need to move beyond our method of using truth tables if we want to pick out all logical equivalences. We will see how to do this in subsequent chapters.

Tautological validity

An argument is valid if it is logically impossible for the premises to be true and for the conclusion to be false at the same time.

As we did for the notion of *logical equivalence*, we can make the notion of *validity* more precise in PL by appealing to a joint truth table for the premises and the conclusion. We say that an argument is TAUTOLOGICALLY VALID if there is no row in the joint truth table that assigns a T to each of the premises and an F to the conclusion.

Consider this argument:

$$\begin{array}{rl} & Oa \vee \big((Hea \vee Oa)\big) \\ & \neg Oa \\ \therefore & Hea \end{array}$$

Is this tautologically valid? Let's draw a joint truth table:

Conclusion *Hea*	*Oa*	(*Hea* ∨ *Oa*)	Premise (*Oa* ∨ (*Hea* ∨ *Oa*))	Premise ¬ *Oa*
T	T	T	T	F
T	F	T	T	T
F	T	T	T	F
F	F	F	F	T

Yes, the argument is tautologically valid. The only row on which both premises receive Ts is the second row, and on that row the conclusion also receives a T. While all tautologically valid arguments can be picked out using our truth table method, it cannot identify all valid arguments. Consider the following argument:

$$\begin{array}{rl} & Bcd \\ & c = e \\ \therefore & Bed \end{array}$$

This is a valid argument: it is logically impossible for the premises to be true and for the conclusion to be false at the same time. Yet, a joint truth table will show that it is not tautologically valid. (We leave it to the reader to show this. Which row is the culprit?) As with logical equivalence, we will need to move beyond our truth table method if we want to pick out all valid arguments.

Summary of logical notions

- A tautology is a sentence whose truth table receives a T in every row of the column under its main connective.
- A contradiction or tautological falsehood is a sentence whose truth table receives an F in every row of the column under its main connective.
- A tautological contingency is a sentence that receives a mixture of Ts and Fs in the column under its main connective.
- Two statements are tautologically equivalent if they have the same entry in every row of their joint truth table.
- An argument is tautologically valid if there is no row in the joint truth table that assigns a T to each of the premises and an F to the conclusion.

Practice exercises

If you want additional practice, you can construct truth tables for any of the sentences and arguments in the exercises for the previous chapter.

Part A Draw truth tables for the following statements. In which cases are you able to say that the statement is a tautology? In which cases are you able to say that it is a contradiction?

1. $(\neg Hab \wedge Hab)$
2. $(\neg Pa \vee Hab)$
3. $(\neg Rcd \vee Rcd)$
4. $((Pa \wedge Hab) \vee (Hab \wedge Pa))$
5. $\Big[[(Pa \wedge Hab) \wedge \neg(Pa \wedge Hab)] \wedge Tc\Big]$
6. $\Big[[(Pa \wedge Hab) \vee (Pa \wedge \neg Hab)] \vee \neg Hab\Big]$

Part B Draw truth tables for each pair of statements. In which cases are you able to say that the statements are tautologically equivalent?

1. Pa, $\neg Pa$
2. Pa, $(Pa \vee Pa)$
3. $\neg(Pa \wedge Hab)$, $(\neg Pa \vee \neg Hab)$
4. $[(Pa \vee Hab) \wedge Tc]$, $[Pa \vee (Hab \wedge Tc)]$

Part C Determine whether each argument is tautologically valid or tautologically invalid. Justify your answer with a joint truth table.

1. $(Pa \land Hab), \quad Tc \quad \therefore \quad (Hab \land Tc)$
2. $(Pa \lor Hab), \quad Tc \quad \therefore \quad (Hab \land Tc)$
3. $(Pa \lor Hab), \quad (Hab \lor Tc), \quad \neg Pa \quad \therefore \quad (Hab \land Tc)$
4. $(Pa \lor Hab), \quad (Hab \lor Tc), \quad \neg Hab \quad \therefore \quad (Pa \land Tc)$

Part D We said in this chapter that a tautology is a sentence whose completed truth table receives a T in every row of the column under the main connective. Similarly, we said that a contradiction is a sentence whose truth table receives an F in every row of the column under its main connective. What does that entail about possibility of atomic tautologies or atomic contradictions?

Chapter 4

Conditionals

4.1 Introducing the conditional

For the following sentences, let *Ru* mean 'You will cut the red wire' and *Eb* mean 'The bomb will explode.'

1. If you cut the red wire, then the bomb will explode.
2. The bomb will explode only if you cut the red wire.

Sentence 1 can be translated partially as 'If *Ru*, then *Eb*.' We will use the symbol '→' to symbolize "If ... then ... " sentences. The sentence becomes $(Ru \rightarrow Eb)$. The connective is called a CONDITIONAL. The sentence on the left-hand side of the conditional (*Ru* in this example) is called the ANTECEDENT. The sentence on the right-hand side (*Eb*) is called the CONSEQUENT.

Sentence 2 is also a conditional. Since the word 'if' appears in the second half of the sentence, it might be tempting to symbolize this in the same way as sentence 1. That would be a mistake.

The conditional $(Ru \rightarrow Eb)$ says that *if Ru* were true, *then Eb* would also be true. It does not say that your cutting the red wire is the *only* way that the bomb could explode. Someone else might cut the wire, or the bomb might be on a timer. The sentence $(Ru \rightarrow Eb)$ does not say anything about what to expect if *Ru* is false. Sentence 2 is different. It says that the only conditions under which the bomb will explode involve your having cut the red wire—i.e., if the bomb explodes, then you must have cut the wire. As such, sentence 2 should be symbolized as $(Eb \rightarrow Ru)$.

It is important to remember that the connective '→' says only that, if the antecedent is true, then the consequent is true. It says nothing about the *causal* connection between the two events. Translating sentence 2 as $(Eb \rightarrow Ru)$ does not mean that the bomb exploding would somehow have caused your cutting the wire. Both sentence 1 and 2 suggest that, if you cut the red wire, your cutting the red wire would be the cause of the bomb exploding. They differ on the *logical* connection. If sentence 2 were true, then an explosion would tell us—those of us safely away from the bomb—that you had cut the red wire. Without an explosion, sentence 2 tells us nothing.

The paraphrased sentence '$\mathcal{P}$ only if $\mathcal{Q}$' is logically equivalent to 'If $\mathcal{P}$, then $\mathcal{Q}$.'

'If $\mathcal{P}$ then $\mathcal{Q}$' means that if $\mathcal{P}$ is true then so is $\mathcal{Q}$. So, we know that if the antecedent $\mathcal{P}$ is true but the consequent $\mathcal{Q}$ is false, then the conditional 'If $\mathcal{P}$ then $\mathcal{Q}$' is false. What is the truth value of 'If $\mathcal{P}$ then $\mathcal{Q}$' under other circumstances? Suppose, for instance, that the antecedent $\mathcal{P}$ happened to be false. 'If $\mathcal{P}$ then $\mathcal{Q}$' would then not tell us anything about the actual truth value of the consequent $\mathcal{Q}$, and it is unclear what the truth value of 'If $\mathcal{P}$ then $\mathcal{Q}$' would be.

In English, the truth of conditionals often depends on what *would* be the case if the antecedent *were true*—even if, as a matter of fact, the antecedent is false. This poses a problem for translating conditionals into TFL. Considered as sentences of TFL, *Ru* and *Eb* in the above examples have nothing intrinsic to do with each other. In order to consider what the world would be like if *Ru* were true, we would need to analyze what *Ru* says about the world. Since *Ru* is an atomic symbol of TFL, however, there is no further structure to be analyzed other than its truth-aptness.

In order to translate conditionals into TFL, we will not try to capture all the subtleties of the English language 'If. . . then. . ..' Instead, the symbol '→' will be a *material conditional*. This means that when $\mathcal{P}$ is false, the conditional $(\mathcal{P}\rightarrow\mathcal{Q})$ is automatically true, regardless of the truth value of $\mathcal{Q}$. If both $\mathcal{P}$ and $\mathcal{Q}$ are true, then the conditional $(\mathcal{P}\rightarrow\mathcal{Q})$ is true.

In short, $(\mathcal{P}\rightarrow\mathcal{Q})$ is false if and only if $\mathcal{P}$ is true and $\mathcal{Q}$ is false. We can summarize this with a characteristic truth table for the conditional:

$\mathcal{P}$	$\mathcal{Q}$	$(\mathcal{P} \rightarrow \mathcal{Q})$
T	T	T
T	F	F
F	T	T
F	F	T

The conditional is *asymmetrical*. You cannot swap the antecedent and consequent without changing the meaning of the sentence, because $(\mathcal{P} \to \mathcal{Q})$ and $(\mathcal{Q} \to \mathcal{P})$ are not logically equivalent.

Not all sentences of the form 'If... then...' are conditionals. Consider this sentence:

3. If anyone wants to see me, then I will be in my office.

If I say this, it means that I will be in my office, regardless of whether anyone wants to see me or not—but if someone did want to see me, then they should look for me there. If we let Ox mean 'x will be in my office' and i mean 'I', then sentence 3 can be translated simply as Oi.

4.2 Introducing the biconditional

Consider these sentences:

4. The figure on the board is a triangle only if it has exactly three sides.
5. The figure on the board is a triangle if it has exactly three sides.
6. The figure on the board is a triangle if and only if it has exactly three sides.

Consider the following symbolization key:

Tx: x is a triangle
Sx: x has three sides
f: the figure on the board

Sentence 4 can be translated as $(Tf \to Sf)$. Sentence 5 is importantly different. It can be paraphrased as 'If the figure has exactly three sides, then the figure on the board is a triangle'. So it can be translated as $(Sf \to Tf)$.

Sentence 6 says that Tf is true *if and only if* Sf is true; we can infer Sf from Tf, and we can infer Tf from Sf. This is called a BICONDITIONAL, because it entails the two conditionals $(Sf \to Tf)$ and $(Tf \to Sf)$. We will use '$\leftrightarrow$' to represent the biconditional; sentence 6 can be translated as $(Tf \leftrightarrow Sf)$.

We could abide without a new symbol for the biconditional. Since sentence 6 means '$(Tf \to Sf)$ and $(Sf \to Tf)$,' we could translate it as $\big((Tf \to Sf) \land (Sf \to Tf)\big)$. Because we could always write $\big((Tf \to Sf) \land (Sf \to Tf)\big)$ instead of $(Tf \leftrightarrow Sf)$, we do not strictly speaking *need* to introduce a new symbol for the biconditional. Indeed, we did not need to introduce a new symbol for the conditional, either. Sentences of the form $(\mathcal{P} \rightarrow \mathcal{Q})$ are logically equivalent to sentences of the form $(\neg\mathcal{P} \vee \mathcal{Q})$, as a joint truth table will make clear. We don't gain expressive power by adding →and ↔in TFL. Conditionals and biconditionals can be perfectly well-expressed simply using the Boolean connectives (i.e., ∧, ∨, ¬). Nevertheless, logical languages usually have symbols for the conditional and the biconditional. TFL will have them, making it easier to translate phrases like 'If. . . then. . .', '. . . only if . . .', and '. . . if and only if . . .'.

$(\mathcal{P} \leftrightarrow \mathcal{Q})$ is true if and only if $\mathcal{P}$ and $\mathcal{Q}$ have the same truth value. This is the characteristic truth table for the biconditional:

$\mathcal{P}$	$\mathcal{Q}$	$(\mathcal{P} \leftrightarrow \mathcal{Q})$
T	T	T
T	F	F
F	T	F
F	F	T

4.3 The truth-functional completeness of the Boolean connectives

We said that we don't gain in expressive power by adding → and ↔ to TFL. Any sentence expressible using → or ↔ can be expressed using ¬, ∧, and ∨. In fact, something else is true of the Boolean connectives. They are *truth-functionally complete*: they can be used to express every truth-functional sentence in TFL. So, we don't need any other *n*-ary connective in TFL.

A full proof of the truth-functional completeness of the Boolean connectives requires a method of proof called 'mathematical induction'. Since we are not familiar with mathematical induction, we are not in a position to offer a complete proof of the truth-functional completeness of the Boolean connectives. But we are in a position to offer a good *sketch* of a proof.

Consider, first, *unary* connectives: connectives that operate on *one* sentence only (e.g., ¬). How can we be certain that there isn't a truth-functional sentence out there whose expression in TFL requires some other unary connective? How do we know that we don't need another unary connective?

Well, suppose we did. Let's use '∗' to name this other unary connective. Note that the characteristic truth table for this unary connective would need to have two (i.e., 2^1) rows. So, it would look something like this:

$\mathcal{P}$	$*\mathcal{P}$
T	1^{st} value
F	2^{nd} value

Since there are two truth values, there are only four (i.e., 2^2) different ways to fill out this truth table:

$\mathcal{P}$	$*\mathcal{P}$
T	T
F	T

(a) Table 1

$\mathcal{P}$	$*\mathcal{P}$
T	T
F	F

(b) Table 2

$\mathcal{P}$	$*\mathcal{P}$
T	F
F	F

(c) Table 3

$\mathcal{P}$	$*\mathcal{P}$
T	F
F	T

(d) Table 4

Table 4.1: Possible characteristic truth tables for '$*$'

If the characteristic truth table for '$*$' was given by table 4.1a, then we could express $*\mathcal{P}$ thusly: $(\mathcal{P} \vee \neg\mathcal{P})$. If the characteristic truth table for '$*$' was given by table 4.1b, we could express $*\mathcal{P}$ with the sentence $\mathcal{P}$. If the characteristic truth table for '$*$' was given by table 4.1c, we could express $*\mathcal{P}$ with $(\mathcal{P} \wedge \neg\mathcal{P})$. And table 4.1d is simply the characteristic truth table for $\neg\mathcal{P}$. So, we are clearly not missing any unary connective.

Now consider *binary* connectives—i.e., connectives that operate on *two* sentences. There are two Boolean binary connectives: $\wedge$ and $\vee$. How can we be certain that there is no truth-functional sentence whose expression requires the use of some other binary connective? Are we missing any binary connective? Again, suppose we were. Let's use '$\spadesuit$' to name this binary connective. Like $\wedge$ and $\vee$, its characteristic truth table would have four (i.e., 2^2) rows. It would look something like this:

$\mathcal{P}$	$\mathcal{Q}$	$(\mathcal{P} \spadesuit \mathcal{Q})$
T	T	1st value
T	F	2nd value
F	T	3rd value
F	F	4th value

Since there are two truth values, there can be a maximum of sixteen (i.e., 4^2) different binary connectives.

Now consider the following conjunctions:

C1: $(\mathcal{P} \wedge \mathcal{Q})$
C2: $(\mathcal{P} \wedge \neg\mathcal{Q})$
C3: $(\neg\mathcal{P} \wedge \mathcal{Q})$
C4: $(\neg\mathcal{P} \wedge \neg\mathcal{Q})$

If the characteristic truth table for $(\mathcal{P} \spadesuit \mathcal{Q})$ assigns a T in the first row only, then $(\mathcal{P} \spadesuit \mathcal{Q})$ is simply conjunction ($\wedge$): C1. If it assigns a T only in the second, third, or fourth row, then we can express $(\mathcal{P} \spadesuit \mathcal{Q})$ with C2, C3, or C4 respectively. If the characteristic truth table for $(\mathcal{P} \spadesuit \mathcal{Q})$ assigns more than one T, then $(\mathcal{P} \spadesuit \mathcal{Q})$ will be a disjunction of the conjunctions that correspond to the row numbers that receive a T. For example, if the characteristic truth table for $(\mathcal{P} \spadesuit \mathcal{Q})$ were:

$\mathscr{P}$	$\mathscr{Q}$	$(\mathscr{P} \spadesuit \mathscr{Q})$
T	T	T
T	F	T
F	T	F
F	F	T

then we could express $(\mathscr{P} \spadesuit \mathscr{Q})$ with C1 $\lor$ C2 $\lor$ C4: $((\mathscr{P} \land \mathscr{Q}) \lor (\mathscr{P} \land \neg\mathscr{Q}) \lor (\neg\mathscr{P} \land \neg\mathscr{Q}))$.

If the characteristic truth table for $(\mathscr{P} \spadesuit \mathscr{Q})$ assigns an F in every row, then we can express $(\mathscr{P} \spadesuit \mathscr{Q})$ thusly: $(\mathscr{P} \land \neg\mathscr{P})$. So we're clearly not missing any binary connective, either.

The same kind of procedure will reveal that we do not need any other *n*-ary truth functional connective. Consider, for example, a ternary connective $\heartsuit$ with the following characteristic truth table:

$\mathscr{P}$	$\mathscr{Q}$	$\mathscr{R}$	$\heartsuit(\mathscr{P}, \mathscr{Q}, \mathscr{R})$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

We can see that $\heartsuit(\mathscr{P}, \mathscr{Q}, \mathscr{R})$ is true in only four circumstances—i.e., when $\mathscr{P}$, $\mathscr{Q}$, and $\mathscr{R}$ are all true (row 1), when $\mathscr{P}$ is true and $\mathscr{Q}$ and $\mathscr{R}$ are false (row 4), when $\mathscr{P}$ is false and $\mathscr{Q}$ and $\mathscr{R}$ are true (row 5), and when $\mathscr{P}$ and $\mathscr{Q}$ are false and $\mathscr{R}$ is true (row 7). So, $\heartsuit(\mathscr{P}, \mathscr{Q}, \mathscr{R})$ can be expressed with the Boolean connectives in the following way:

$$\begin{gathered}\Big(\Big[\big(\mathscr{P} \land (\mathscr{Q} \land \mathscr{R})\big) \lor \big(\mathscr{P} \land (\neg\mathscr{Q} \land \neg\mathscr{R})\big)\Big] \\ \lor \\ \Big[\big(\neg\mathscr{P} \land (\mathscr{Q} \land \mathscr{R})\big) \lor \big(\neg\mathscr{P} \land (\neg\mathscr{Q} \land \mathscr{R})\big)\Big]\Big)\end{gathered}$$

Granted, this is a fairly long and cumbersome disjunction. But it shows that we don't need to add the ternary connective $\heartsuit$ in TFL.

This is only a *sketch* of a proof that the Boolean connectives are truth functionally complete. Once we've established the truth-functional completeness of the Boolean connectives, we can also show that the sets of connectives $\{\neg, \land\}$ and $\{\neg, \lor\}$ are truth-functionally complete on their own.

To see this, consider first De Morgan's Laws, two valid transformation rules named after 19th century British logician Augustus De Morgan:

$$\text{DM1: } \neg(\mathcal{P} \wedge \mathcal{Q}) \Leftrightarrow (\neg\mathcal{P} \vee \neg\mathcal{Q})$$
$$\text{DM2: } \neg(\mathcal{P} \vee \mathcal{Q}) \Leftrightarrow (\neg\mathcal{P} \wedge \neg\mathcal{Q})$$

Next, consider another valid transformation rule that we might call 'double negation':

$$\text{DN: } \mathcal{P} \Leftrightarrow \neg\neg\mathcal{P}$$

DN simply states that any sentence is logically equivalent to the negation of its negation. (Joint truth tables will show that DM1, DM2, and DN state logical equivalences. You are encouraged to do them.)

DM1, DM2, and DN can be used to show how to express $\wedge$ using $\neg$ and $\vee$, and how to express $\vee$ using $\neg$ and $\wedge$, thus showing that if $\{\neg, \wedge, \vee\}$ is truth-functionally complete, so are $\{\neg, \wedge\}$ and $\{\neg, \vee\}$:

$$(\mathcal{P} \wedge \mathcal{Q}) \Leftrightarrow \neg(\neg\mathcal{P} \vee \neg\mathcal{Q})$$
$$(\mathcal{P} \vee \mathcal{Q}) \Leftrightarrow \neg(\neg\mathcal{P} \wedge \neg\mathcal{Q})$$

4.4 Unless

We have now introduced all the connectives of TFL: $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$. We can use them together to translate many kinds of sentences. Consider the examples of sentences that use the English-language connective 'unless':

7. Unless you wear a jacket, you will catch a cold.
8. You will catch a cold unless you wear a jacket.

Now take the following symbolization key:

Jx: x wears a jacket
Cx: x will catch a cold
u: You

We can paraphrase sentence 7 as 'Unless Ju, Cu.' This means that if you do not wear a jacket, then you will catch cold; with this in mind, we might translate it as $(\neg Ju \rightarrow Cu)$. It also means that if you do not catch a cold, then you must have worn a jacket; with this in mind, we might translate it as $(\neg Cu \rightarrow Ju)$.

Which of these is the correct translation of sentence 7? Both translations are correct, because the two translations are logically equivalent in TFL. A joint truth table will reveal this. (Again, you are encouraged to do one.)

Sentence 8, in English, is logically equivalent to sentence 7. So, it too can be translated as $(\neg Ju \rightarrow Cu)$ or $(\neg Cu \rightarrow Ju)$.

When symbolizing sentences like sentence 7 and sentence 8, it is easy to get turned around. Since the conditional is not symmetric, it would be wrong to translate either sentence as $(Ju \rightarrow \neg Cu)$. Fortunately, there are other logically equivalent expressions. Both sentences mean that you will wear a jacket or you will catch a cold (if you don't wear a jacket). So we can in effect translate them as $(Ju \vee Cu)$. (You might worry that the 'or' here should be an *exclusive or*. However, the sentences do not exclude the possibility that you might *both* wear a jacket *and* catch a cold; jackets do not protect you from all the possible ways that you might catch a cold.)

If a sentence can be paraphrased as 'Unless $\mathcal{P}$, $\mathcal{Q}$,' then it can be symbolized as $(\mathcal{P} \vee \mathcal{Q})$.

Summary of logical notions

- A CONDITIONAL sentence is true if and only if the antecedent is false or the consequent is true.
- The ANTECEDENT is the left-hand sentence of a conditional sentence of the form 'if ... then ...'.
- The CONSEQUENT is the right-hand sentence of a conditional sentence of the form 'if ... then ...'.
- A BICONDITIONAL sentence is true if and only if both the left- and right-hand sentences have the same truth value.

Practice exercises

Part A Using the symbolization key given, translate each English-language sentence into TFL.

Mx: x is a man in a suit
Cx: x is a chimpanzee
Gx: x is a gorilla
b: Bob
k: Koko
f: Flo

1. If Flo is a chimpanzee, then she is not a gorilla.
2. If Koko is not a man in a suit, then she's either a chimpanzee or a gorilla.
3. If Bob is a chimpanzee, then he is neither a gorilla nor a chimpanzee.
4. Unless Bob is a man in a suit, he is either a chimpanzee or a gorilla.

Part B Using the symbolization key given, translate each English-language sentence into TFL. The translations build off each other, such that something established in an earlier translation is sometimes used in subsequent sentences.

M_1x: x was murdered
M_2xy: x murdered y
Lx: x is lying
Fx: x was a frying pan
a: Mister Ace
b: the butler
c: the cook
d: the Duchess
e: Mister Edge
w: the murder weapon

1. If Mister Ace was murdered, then the cook did it.
2. If Mister Edge was murdered, then the cook did not do it.
3. The cook did it only if the Duchess is lying.
4. If the murder weapon was a frying pan, then the culprit must have been the cook.
5. If the murder weapon was not a frying pan, then the culprit was either the cook or the butler.
6. Mister Ace was murdered if and only if Mister Edge was not murdered.
7. The Duchess is lying, unless it was Mister Edge who was murdered.
8. If Mister Ace was murdered, he was murdered with a frying pan.

Part C Using the symbolization key given, translate each English-language sentence into TFL.

Ex: x is an electrician.
Fx: x is a firefighter.
Sx: x is satisfied with their career.
a: Ava
h: Harrison

1. If Ava is a firefighter, then she is satisfied with her career.
2. Ava is a firefighter, unless she is an electrician.
3. Harrison is satisfied only if he is firefighter.
4. If Ava is not an electrician, then neither is Harrison, but if she is, then he is too.
5. Ava is satisfied with her career if and only if Harrison is not satisfied with his.
6. If Harrison is both an electrician and a firefighter, then he must be satisfied with his work.
7. Harrison and Ava are both firefighters if and only if neither of them is an electrician.

Part D Give a symbolization key and symbolize the following sentences in TFL.

1. Alice and Bob are both spies.
2. If either Alice or Bob is a spy, then the code has been broken.
3. If neither Alice nor Bob is a spy, then the code remains unbroken.
4. The German embassy will be in an uproar, unless someone has broken the code.
5. Either the code has been broken or it has not, but the German embassy will be in an uproar regardless.
6. Either Alice or Bob is a spy, but not both.

Part E For each argument, write a symbolization key and translate the argument as well as possible into TFL.

1. If Dorothy plays the piano in the morning, then Roger wakes up cranky. Dorothy plays piano in the morning unless she is distracted. So if Roger does not wake up cranky, then Dorothy must be distracted.
2. The precipitation is either rain or snow. If it is rain, then Neville will be sad. If it is snow, then Neville will be cold. Therefore, Neville will either be sad or cold.
3. If Zoog remembered to do his chores, then things are clean but not neat. If he forgot, then things are neat but not clean. Therefore, things are either neat or clean—but not both.

Part F Determine whether each argument is valid or invalid.

1. $(Pa \rightarrow Qa), \quad Pa, \quad \therefore \quad Qa$
2. $(Pa \rightarrow Qa), \quad Qa, \quad \therefore \quad Pa$
3. $(Pa \rightarrow Qa), \quad \neg Qa, \quad \therefore \quad \neg Pa$
4. $(Pa \rightarrow Qa), \quad \neg Pa, \quad \therefore \quad \neg Qa$
5. $(Pa \leftrightarrow Qa), \quad \therefore \quad (Qa \leftrightarrow Pa)$
6. $(Pa \leftrightarrow Qa), \quad \therefore \quad (\neg Pa \leftrightarrow \neg Qa)$
7. $(Pa \leftrightarrow Qa), \quad \therefore \quad (Pa \vee \neg Qa)$
8. $(Pa \rightarrow Qa), \quad \therefore \quad (\neg Qa \rightarrow \neg Pa)$
9. $(Pa \rightarrow Qa), \quad \therefore \quad (\neg Pa \rightarrow \neg Qa)$
10. $(Pa \rightarrow Qa), \quad (Qa \rightarrow Ra), \quad \therefore \quad (Pa \rightarrow Ra)$
11. $(Pa \vee (Qa \rightarrow Pa)), \quad \therefore \quad (\neg Pa \rightarrow \neg Qa)$

Part G Given that the Boolean connectives are truth-functionally complete, show that the sets $\{\neg, \wedge\}$ and $\{\neg, \vee\}$ are also truth-functionally complete.

Part H Show how to express the following ternary truth functional connective using just the Boolean connectives.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{R}$	$\clubsuit(\mathcal{P}, \mathcal{Q}, \mathcal{R})$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

Chapter 5

Introducing proofs

Consider these two arguments in TFL:

Argument A	Argument B
$(Kc \vee Rca)$	$(Kc \to Rca)$
$\neg Kc$	Kc
$\therefore\ Rca$	$\therefore\ Rca$

Clearly, these are valid arguments. Argument A makes use of an inference form that is always valid: Given a disjunction and the negation of one of the disjuncts, the other disjunct follows as a valid consequence. This rule is called *disjunctive syllogism*, and you can confirm that the corresponding argument is valid by constructing a four-line joint truth table.

Argument B makes use of a different valid form: Given a conditional and its antecedent, the consequent follows as a valid consequence. This is called *modus ponens*. Again, you can confirm that the corresponding argument is valid by constructing a four-line joint truth table.

Using joint truth tables to check for validity has its limits. Consider, for example, the number of rows in a joint truth table we would need to check for the validity of an argument containing six different atomic sentences: 64 (i.e., 2^6) rows! Even if we were willing to take the time and meticulously fill out such a joint truth table, we have seen in the previous chapter that joint truth tables can only check for *tautological validity*. But as we have seen, some arguments are valid yet not tautologically so. Consider the following argument:

Argument C

$$\begin{array}{ll} & Pa \\ & a = b \\ \therefore & Pb \end{array}$$

This argument is clearly valid, yet a joint truth table will not show this. There is one row of the joint truth table that assigns Ts to both Pa and $a = b$ yet an F to Pb. So we need a more sophisticated method that can prove the validity of this argument.

Consider now this inference:

Premise One: $\big(\neg Td \rightarrow (Rad \vee Td)\big)$
Premise Two: $\neg Td$
Conclusion: Rad

By *modus ponens*, Premise One and Premise Two entail $(Rad \vee Td)$. This is an *intermediate conclusion*. It follows logically from the premises, but it is not the conclusion we want. Now $(Rad \vee Td)$ and Premise Two entail Rad, by disjunctive syllogism, which is the desired conclusion.

This argument just described can be presented like this:

1	$\big(\neg Td \rightarrow (Rad \vee Td)\big)$	
2	$\neg Td$	
3	$(Rad \vee Td)$	By *modus ponens*, from lines 1 and 2
4	Rad	By disjunctive syllogism, from lines 2 and 3

This is a PROOF. It is a sequence of statements. The premises come first; the desired conclusion is at the end. Each statement that is not a premise is justified by appeal to some inference rule, applied to statements earlier in the sequence.

We write the proof as a series of numbered lines. The premise or premises come first. We draw a horizontal line to separate the premise(s) from the later statements. For each later statement, we explain how the statement is justified, citing the rule and the earlier lines which have been used. These explanatory comments are written in a column on the right-hand side. The final sentence is the conclusion of the argument.

So far, we have only considered two rules: *modus ponens* and disjunctive syllogism. This isn't enough. If we want to be able to prove every logical consequence that our joint truth table method could prove, we need more rules. In the rest of this chapter and in the following one, we will present more.

In designing a proof system—i.e., a system of inference rules—we could just start with disjunctive syllogism and *modus ponens*. Whenever we discovered a valid argument which could not be proven with rules we already had, we could introduce new rules. Proceeding in this way, we would have an unsystematic grab bag of rules. We might accidentally add some strange rules, and we would surely end up with more rules than we need.

Instead, we will develop what is called a natural deduction system. In a natural deduction system, there will be two rules for each logical operator: an introduction rule that allows us to prove a sentence that has it as the main logical operator and an elimination rule that allows us to prove something given a sentence that has it as the main logical operator. In TFL, the main logical operators are the truth-functional connectives.

In TFL, we treat the identity symbol as a logical symbol with a strict interpretation—i.e., it always expresses the identity relation. Because of this, we will also have introduction and elimination rules for identity. Finally, we will have a reiteration rule. If you have shown something in the course of a proof, the reiteration rule allows you to repeat it on a new line. Here is the reiteration rule, which we abbreviate (R):

$$\begin{array}{r|ll} j & \mathcal{P} & \\ k & \mathcal{P} & \text{R } j \end{array}$$

When we add a line to a proof, we write the rule that justifies that line. We also write the numbers of the lines to which the rule was applied. The reiteration rule above is justified by one line, the line that you are reiterating. The 'R *j*' on line *k* of the proof means that the line is justified by the reiteration rule (R) applied to line *j*. Of course, '*j*' and '*k*' are not numbers. Rather, they *stand* for line numbers in a proof. When we define the rule, we use letters like '*j*' and '*k*' to underscore the point that the rule may be applied at any step in a proof. If you have *Pa* on line 8, you can reiterate it on line 15 at some later point in the proof with the justification 'R 8'.

Obviously, the reiteration rule will not allow us to show anything *new*. For that, we will need more rules. In the coming section, we consider the rules for conjunction. In subsequent sections, we consider the rules for disjunction and identity, as well as one rule for the conditional.

5.1 Rules for conjunction

Think for a moment: What would you need to show in order to prove $(Rac \wedge Th)$? Of course, you could show $(Rac \wedge Th)$ by proving Rac and separately proving Th. This holds even if the two conjuncts are not atomic sentences. If you can prove $(Pb \vee a = c)$ and $(Uea \vee Oa)$, then you have effectively proven that

$\big((Pb \vee a = c) \wedge (Uea \vee Oa)\big)$.

So, this will be our CONJUNCTION INTRODUCTION rule, which we abbreviate (∧I):

j	$\mathcal{P}$	
k	$\mathcal{Q}$	
l	$(\mathcal{P} \wedge \mathcal{Q})$	$\wedge$I j, k

A line of proof must be justified by some rule, and here we have '∧I j, k.' This means: Conjunction introduction applied to line j and line k, where 'j' and 'k' stand for real line numbers.

With the ∧I rule, P and Q can be any two statements—even complicated ones! For example, we might use the rule like this:

	$\dots$	
	$\dots$	
23	$(Rad \to Bh)$	
	$\dots$	
	$\dots$	
34	$(a = b \leftrightarrow a = c)$	
	$\dots$	
	$\dots$	
56	$\big((Rad \to Bh) \wedge (a = b \leftrightarrow a = c)\big)$	$\wedge$I 23, 34

Now, consider the elimination rule for conjunction. What are you entitled to conclude from a sentence like $(Rad \wedge Bh)$? Surely, you are entitled to conclude Rad; if $(Rad \wedge Bh)$ were true, then Rad would be true. Similarly, you are entitled to conclude Bh. This will be our CONJUNCTION ELIMINATION rule, which we abbreviate (∧E):

j	$(\mathcal{P} \wedge \mathcal{Q})$	
k	$\mathcal{P}$	$\wedge$E j

j	$(\mathcal{P} \wedge \mathcal{Q})$	
k	$\mathcal{Q}$	$\wedge$E j

When you have a conjunction on some line of a proof, you can use ∧E to derive either of the conjuncts. The ∧E rule requires only one sentence, so we write one line number as the justification for applying it.

Even with just these two rules, we can provide some proofs. Consider this argument.

$$\begin{array}{rl} & (Rad \wedge Bh) \\ \therefore & (Bh \wedge Rad) \end{array}$$

The main logical operator in both the premise and conclusion is conjunction. Since conjunction is symmetric, the argument is obviously valid. In order to provide a proof, we begin by writing down the premise or premises. After the premise(s), we draw a horizontal line—everything below this line must be justified by a rule of proof.

So the beginning of the proof looks like this:

1	$(Rad \wedge Bh)$	

From the premise, we can get each of the conjuncts by ∧E. The proof now looks like this:

1	$(Rad \wedge Bh)$	
2	Rad	∧E 1
3	Bh	∧E 1

The rule ∧I requires that we have each of the conjuncts available somewhere in the proof. They can be separated from one another, and they can appear in any order. So by applying the ∧I rule to lines 3 and 2, we arrive at the desired conclusion. The finished proof looks like this:

1	$(Rad \wedge Bh)$	
2	Rad	∧E 1
3	Bh	∧E 1
4	$(Bh \wedge Rad)$	∧I 3, 2

This proof is trivial, but it shows how we can use rules of proof together to demonstrate the validity of an argument form.

5.2 Rules for disjunction

The DISJUNCTION INTRODUCTION rule (∨I) allows us to derive a disjunction if we have one of the two disjuncts:

$$\begin{array}{r|ll} j & \mathcal{P} & \\ k & (\mathcal{P} \lor \mathcal{Q}) & \lor\text{I } j \end{array} \qquad\qquad \begin{array}{r|ll} j & \mathcal{Q} & \\ k & (\mathcal{P} \lor \mathcal{Q}) & \lor\text{I } j \end{array}$$

This is obviously a valid rule of inference. All it takes for a disjunction to be true is for at least one of its conjuncts to be true. So, if $\mathcal{P}$ (or $\mathcal{Q}$) were true, then $(\mathcal{P} \lor \mathcal{Q})$ would also be true.

Notice that the sentence you are introducing as the second disjunct can be *any* sentence whatsoever. So, the following is a legitimate proof:

$$\begin{array}{r|ll} 1 & Rad & \\ \cline{2-2} 2 & \big(Rad \lor (Tu \leftrightarrow a = c)\big) & \lor\text{I } 1 \end{array}$$

It may seem odd that just by knowing *Rad* we can derive a conclusion that includes sentences like Tu and $a = c$—sentences that have nothing to do with Rad. Yet the conclusion follows immediately by ∨I. This is as it should be: The truth conditions for the disjunction mean that, if P_1n_1 is true, then $(P_1n_1 \lor P_2n_2)$ is true regardless of what P_2n_2 is. So, the conclusion could not be false if the premise were true; the argument is valid.

Now consider the disjunction elimination rule. What can you conclude from $(Rad \lor Bh)$? You cannot conclude *Rad*. It might be *Rad*'s truth that makes the disjunction true, as in the example above, but it might not. From $(Rad \lor Bh)$ alone, you cannot conclude anything about either *Rad* or *Bh* specifically. If you also knew that *Bh* was false, however, then you would be able to conclude *Rad*.

This is just disjunctive syllogism, it will be the disjunction elimination rule (∨E).

$$\begin{array}{r|ll} j & (\mathcal{P} \lor \mathcal{Q}) & \\ k & \neg\mathcal{Q} & \\ l & \mathcal{P} & \lor\text{E } j, k \end{array} \qquad\qquad \begin{array}{r|ll} j & (\mathcal{P} \lor \mathcal{Q}) & \\ k & \neg\mathcal{P} & \\ l & \mathcal{Q} & \lor\text{E } j, k \end{array}$$

Let's put the rules for disjunction to work in a longer proof. Let's prove the following inference:

$$\begin{array}{ll} & (Rad \lor Bh) \\ & \neg Rad \\ \therefore & \big(Bh \lor (Tu \to a = c)\big) \end{array}$$

As usual, we begin by writing down our premises or premises, followed by a horizontal line:

1	$(Rad \lor Bh)$	
2	$\neg Rad$	

The first thing we notice is that we can prove *Bh* by disjunctive syllogism, which is also called '∨E':

1	$(Rad \lor Bh)$	
2	$\neg Rad$	
3	Bh	∨E 1, 2

From *Bh* we can infer $\big(Bh \lor (Tu \to a = c)\big)$, which is the conclusion we ultimately want. We are now able to complete the proof:

1	$(Rad \lor Bh)$	
2	$\neg Rad$	
3	Bh	∨E 1, 2
4	$\big(Bh \lor (Tu \to a = c)\big)$	∨I 3

5.3 A rule for conditionals

The introduction rule for the conditional is difficult; we will set it aside until the next chapter. For now, we will use only the elimination rule.

Nothing follows from $(Rad \to Bh)$ alone, but if we have both $(Rad \to Bh)$ and *Rad*, then we can conclude *Bh*. This rule, *modus ponens*, will be the CONDITIONAL ELIMINATION rule (→E).

j	$(\mathcal{P} \to \mathcal{Q})$	
k	$\mathcal{P}$	
l	$\mathcal{Q}$	→E j, k

Let's put this rule to work in a longer proof. Suppose we wish to prove the following inference:

$(Ra \to Rb)$
$(Rb \to Rc)$
Ra

∴ Rc

This can be done in two steps. The first conditional gets us from Ra to Rb. The second conditional gets us from Rb to Rc. This is what it looks like when written out in the proper format:

1	$(Ra \to Rb)$	
2	$(Rb \to Rc)$	
3	Ra	
4	Rb	$\to$E 1, 3
5	Rc	$\to$E 2, 4

5.4 Rules for identity

Here is a logical truth: everything is identical to itself. For any object n, n is identical to n. This is called the '*reflexivity of identity*'. Here is an example: '$1 = 1$'. Because '$1 = 1$' is a logical truth, it is impossible for it to be false. And since it is impossible for '$1 = 1$' to be false, it is a logical consequence of any sentence. So, we are justified to introduce a corresponding introduction rule that will allow us to introduce identity statements like '$1 = 1$' or '$a = a$' at any line in a proof. Such a rule will never lead us to an invalid step. (Remember: we stipulated that individual constants uniquely refer.) So, this will be our identity introduction rule, which we abbreviate (=I):

j	$n = n$	=I

As we said earlier line of a proof (other than the premises) must be justified by some rule. Here, on line j, we have the rule '=I'. We also said earlier that we need to cite the line number(s) on which the rule is applied. But we make an exception with (=I). Since logical truths follow from any sentence, we do not need to cite any line number when using the rule.

Next, consider what is called the *indiscernibility of identicals*. According to this principle, if two things are identical, then whatever is true of one is also true of the other. This is a logical truth. It is impossible for it to be false. If there is something that is true of one but not the other, then those two things are not identical. This means that we are justified to introduce the corresponding identity elimination rule, which we abbreviate (=E):

$$\begin{array}{r|ll}
j & \mathcal{P}n & \\
k & n = m & \\
l & \mathcal{P}m & =\text{E } j, k
\end{array}$$

Line *j* tells us that some object named *n* has the property $\mathcal{P}$. Line *k* tells us that the object named by '*n*' is identical to the object named by '*m*'. Line *l* tells us that, given lines *j* and *k*, and by the principle of the indiscernibility of identicals (=E), *m* also has the property $\mathcal{P}$.

Now that we have the rule for =E, we are now in a position to prove the validity of argument C above, an argument whose validity couldn't be established using the method of truth tables.

$$\begin{array}{r|ll}
1 & Pa & \\
2 & a = b & \\ \cline{2-2}
3 & Pb & =\text{E } 1, 2
\end{array}$$

Here is a proof that uses both rules, showing that $b = a$ follows from $a = b$.

$$\begin{array}{r|ll}
1 & a = b & \\ \cline{2-2}
2 & a = a & =\text{I} \\
3 & b = a & =\text{E } 1, 2
\end{array}$$

According to the premise, *a* is identical to *b*. This means that *a* and *b* share the same properties. Line 2 states that *a* has the property of being identical to *a*, which is justified by the reflexivity of identity. So, by the indiscernibility of identicals, *b* has the property of being identical to *a*, which is our desired conclusion.

5.5 Three more complex examples

Suppose we wish to prove the following inference:

$$\begin{array}{rl}
 & (Rac \land Ue) \\
 & (Rac \rightarrow Fah) \\
 & (Ue \rightarrow Kai) \\
\therefore & (Fah \land Kai)
\end{array}$$

We can begin by writing down our premises:

1	$(Rac \wedge Ue)$	
2	$(Rac \rightarrow Fah)$	
3	$(Ue \rightarrow Kai)$	

Now where should we begin? There's nothing we can do with the conditionals at the moment. But we can split the conjunction into its conjuncts, using the (∧E) rule. This gives us:

1	$(Rac \wedge Ue)$	
2	$(Rac \rightarrow Fah)$	
3	$(Ue \rightarrow Kai)$	
4	Rac	∧E 1
5	Ue	∧E 1

Now that we've proven *Rac* and *Ue*, we can use (→E) twice, to prove *Fah* and *Kai*:

1	$(Rac \wedge Ue)$	
2	$(Rac \rightarrow Fah)$	
3	$(Ue \rightarrow Kai)$	
4	Rac	∧E 1
5	Ue	∧E 1
6	Fah	→E 2, 4
7	Kai	→E 3, 5

Now that we've proven *Fah* and *Kai*, we can get their conjunction by (∧I), and this completes the proof:

1	$(Rac \wedge Ue)$	
2	$(Rac \rightarrow Fah)$	
3	$(Ue \rightarrow Kai)$	
4	Rac	$\wedge$E 1
5	Ue	$\wedge$E 1
6	Fah	$\rightarrow$E 2, 4
7	Kai	$\rightarrow$E 3, 5
8	$(Fah \wedge Kai)$	$\wedge$I 6, 7

Suppose we wish to prove the following inference:

Rea

$\big(Rea \rightarrow (b = c \vee Rae)\big)$

$b \neq c$

$\therefore$ $(Rea \wedge Rae)$

The conclusion is a conjunction, so it's a good bet that we'll prove it using the $\wedge$I rule. So, we want to prove two conjuncts. That is, we want to prove *Rea* and *Rae*. Now, *Rea* is one of our premises, so no work is required there! However, we do need to prove *Rae*, which is trickier. The first two premises give us $(b = c \vee Rae)$ by $\rightarrow$E, which looks to be a good start:

1	Rea	
2	$\big(Rea \rightarrow (b = c \vee Rae)\big)$	
3	$b \neq c$	
4	$(b = c \vee Rae)$	$\rightarrow$E 1, 2

Now we notice that lines 3 and 4 get us *Rae* by $\vee$E (remember that $b \neq c$ is just shorthand for $\neg b = c$). From there, it is a short step to our desired conclusion:

1	Rea	
2	$\big(Rea \to (b = c \vee Rae)\big)$	
3	$b \neq c$	
4	$(b = c \vee Rae)$	$\to$E 1, 2
5	Rae	$\vee$E 4, 3
6	$(Rea \wedge Rae)$	$\wedge$I 1, 5

Let's finish the chapter with one more example. Let's prove the following inference:

$$Tu$$

$$(Tu \to Laa)$$

$$\big(Laa \to (u = b \vee Lba)\big)$$

$$\neg Lba$$

$$\therefore \big(a = a \wedge (b = b \vee Tb)\big)$$

We can see that two applications of →E can get us $(u = b \vee Lba)$:

1	Tu	
2	$(Tu \to Laa)$	
3	$\big(Laa \to (u = b \vee Lba)\big)$	
4	$\neg Lba$	
5	Laa	$\to$E 1, 2
6	$(u = b \vee Lba)$	$\to$E 3, 5

We can now see that line 4 is just the negation of one of the disjuncts on line 6. So, by disjunctive syllogism we get $u = b$:

1	Tu	
2	$(Tu \to Laa)$	
3	$\big(Laa \to (u = b \lor Lba)\big)$	
4	$\neg Lba$	
5	Laa	→E 1, 2
6	$(u = b \lor Lba)$	→E 3, 5
7	$u = b$	∨E 6, 4

Now with the ∨I rule we get one of the conclusion's conjuncts:

1	Tu	
2	$(Tu \to Laa)$	
3	$\big(Laa \to (u = b \lor Lba)\big)$	
4	$\neg Lba$	
5	Laa	→E 1, 2
6	$(u = b \lor Lba)$	→E 3, 5
7	$u = b$	∨E 6, 4
8	$(u = b \lor Tb)$	∨I 7

Finally, we can get $a = a$ with an application of =I, and our desired conclusion is a short step away:

1	Tu	
2	$(Tu \to Laa)$	
3	$\big(Laa \to (u = b \lor Lba)\big)$	
4	$\neg Lba$	
5	Laa	$\to$E 1, 2
6	$(u = b \lor Lba)$	$\to$E 3, 5
7	$u = b$	$\lor$E 6, 4
8	$(u = b \lor Tb)$	$\lor$I 7
9	$a = a$	=I
10	$\big(a = a \land (u = b \lor Tb)\big)$	$\land$I 9, 8

Summary of logical notions

- ▹ A PROOF is a step-by-step demonstration that one statement (i.e., the conclusion) is a logical consequence of other statements (i.e., the premises).
- ▹ A NATURAL DEDUCTION SYSTEM consists in a fixed set of rules that are used to construct formal proofs.

Summary of derivation rules in TFL covered in this chapter

REITERATION (R)

j	$\mathcal{P}$	
k	$\mathcal{P}$	R j

CONJUNCTION INTRODUCTION ($\land$I)

j	$\mathcal{P}$	
k	$\mathcal{Q}$	
l	$(\mathcal{P} \land \mathcal{Q})$	$\land$I j, k

Conjunction Elimination (∧E)

$$\begin{array}{r|ll} j & (\mathcal{P} \wedge \mathcal{Q}) & \\ k & \mathcal{P} & \wedge\text{E } j \end{array} \qquad\qquad \begin{array}{r|ll} j & (\mathcal{P} \wedge \mathcal{Q}) & \\ k & \mathcal{Q} & \wedge\text{E } j \end{array}$$

Disjunction Introduction (∨I)

$$\begin{array}{r|ll} j & \mathcal{P} & \\ k & (\mathcal{P} \vee \mathcal{Q}) & \vee\text{I } j \end{array} \qquad\qquad \begin{array}{r|ll} j & \mathcal{Q} & \\ k & (\mathcal{P} \vee \mathcal{Q}) & \vee\text{I } j \end{array}$$

Disjunction Elimination (∨E)

$$\begin{array}{r|ll} j & (\mathcal{P} \vee \mathcal{Q}) & \\ k & \neg\mathcal{Q} & \\ l & \mathcal{P} & \vee\text{E } j, k \end{array} \qquad\qquad \begin{array}{r|ll} j & (\mathcal{P} \vee \mathcal{Q}) & \\ k & \neg\mathcal{P} & \\ l & \mathcal{Q} & \vee\text{E } j, k \end{array}$$

Conditional Elimination (→E)

$$\begin{array}{r|ll} j & (\mathcal{P} \to \mathcal{Q}) & \\ k & \mathcal{P} & \\ l & \mathcal{Q} & \to\text{E } j, k \end{array}$$

Identity Introduction (=I)

$$\begin{array}{r|ll} j & \mathcal{n} = \mathcal{n} & =\text{I} \end{array}$$

Identity Elimination (=E)

$$\begin{array}{r|ll} j & \mathcal{P}\mathcal{n} & \\ k & \mathcal{n} = \mathcal{m} & \\ l & \mathcal{P}\mathcal{m} & =\text{E } j, k \end{array}$$

Practice exercises

Part A [*] Identify the mistake in each of the following "proofs".

1.

$$\begin{array}{r|ll}
1 & (Pa \land \neg Qb) & \\
2 & \big(Pa \to (Qb \land Rc)\big) & \\
3 & (Qb \land Rc) & \\
\hline
4 & Pa & \to\text{E } 2, 3
\end{array}$$

2.

$$\begin{array}{r|ll}
1 & Qee & \\
2 & \big((Qee \lor Rmn) \to Pa\big) & \\
\hline
3 & (Qee \lor Rmn) & \lor\text{I } 1 \\
4 & Pa & \to\text{E } 2, 3 \\
5 & \big((Qee \lor Rmn) \lor Pa\big) & \lor\text{I } 3, 4
\end{array}$$

3.

$$\begin{array}{r|ll}
1 & Hn & \\
2 & (Kb \land Rm) & \\
\hline
3 & \big((Hn \land Kb) \land Rm\big) & \land\text{I } 1, 2
\end{array}$$

---

[*] Many thanks to Kesavan Thanagopal for offering these exercises, originally created for his PHIL110 *Introduction to Logic and Reasoning* tutorials.

4.

1	$r = s$	
2	$\big(r = s \to (\neg Mp \lor Cs)\big)$	
3	Mp	
4	$(\neg Mp \lor Cs)$	$\to$E 1, 2
5	$\neg\neg Mp$	From 3
6	Cs	$\lor$E 4, 5
7	$(r \neq s \lor Cs)$	$\lor$I 6

Part B Here are four incomplete proofs. In each case, provide a justification (rule and line numbers) for each line of proof that requires one.

1.

1	$(Qee \land Ga)$
2	$(Hb \land Rde)$
3	Ga
4	Hb
5	$(Ga \land Hb)$

2.

1	$(Hb \land Rde)$
2	Hb
3	$(Hb \lor a = b)$

3.

1	$(Qee \to \neg Ga)$
2	$(Hb \land Qee)$
3	$\big(Ga \lor (Gb \land Gc)\big)$
4	Qee
5	$\neg Ga$
6	$(Gb \land Gc)$
7	Gc

4.

1	$(a \neq b \to a = c)$
2	$\big(Pa \land (a = b \lor Rad)\big)$
3	$a \neq b$
4	Pa
5	$(a = b \lor Rad)$
6	Rad
7	$a = c$
8	Pc
9	Rcd
10	$(Pc \land Rcd)$
11	$b = b$
12	$\big(b = b \land (Pc \land Rcd)\big)$

Part C Here are six inferences. In each case, show that the inference is valid by providing a proof.

1. Ba
   $\big(Ba \to (Cga \land De)\big)$
   $\therefore\ Cga$

2. $\big((Tea \lor Mai) \to Uc\big)$
   Tea
   $\therefore\ Uc$

3. Pa
   Qg
   $\therefore\ \big((Pa \land Qg) \lor Tjk\big)$

4. $\big((He \land Sea) \lor See\big)$
   $\big(Gb \to \neg(He \land Sea)\big)$
   $(Gb \land He)$
   $\therefore\ (He \land See)$

5. $d \neq f$
   $(a = b \lor d = f)$
   $(a = b \to Tab)$
   $(Tbb \to \neg Hu)$
   $(Hu \lor Hi)$
   $\therefore\ (Hi \land c = c)$

6. $\big((Te \lor Ga) \lor Lo\big)$
   $\big(Rbc \to \neg(Te \lor Ga)\big)$
   $\big(Lo \to (\neg Te \land \neg Ga)\big)$
   Raa
   $a = b$
   $b = c$
   $\therefore\ \neg Ga$

Chapter 6

Proofs involving conditionals and negation

6.1 Conditionals

You already know the elimination rule for the conditional. In this section, we discuss the introduction rule.

Consider this argument:

$$\begin{array}{ll} & (Te \lor Ra) \\ \therefore & (\neg Te \to Ra) \end{array}$$

The argument is certainly a valid one. What should the conditional introduction rule be, such that we can draw this conclusion?

We begin the proof by writing down the premise of the argument and drawing a horizontal line, like this:

$$\begin{array}{l|l} 1 & \underline{(Te \lor Ra)} \end{array}$$

If we had $\neg Te$ as a further premise, we could derive Ra by the $\lor$E rule. We do not have $\neg Te$ as a premise of this argument, nor can we derive it directly from the premise we do have—so we cannot simply prove Ra. What we will do instead is start a *subproof*, a proof within the main proof. When we start a subproof, we draw another vertical line to indicate that we are no longer in the main proof. Then we write in an assumption for the subproof. This can be anything we want. Here, it will be helpful to assume $\neg Te$. Our proof now looks like this:

$$\begin{array}{r|lll}
1 & (Te \vee Ra) & & \\ \hline
2 & \quad \vert\ \underline{\neg Te} & &
\end{array}$$

It is important to notice that we are not claiming to have proven $\neg Te$. We do not need to write in any justification for the assumption line of a subproof. You can think of the subproof as posing the question: What could we show *if* $\neg Te$ were true? For one thing, we can derive Ra. So we do:

$$\begin{array}{r|lll}
1 & (Te \vee Ra) & & \\ \hline
2 & \quad \vert\ \underline{\neg Te} & & \\
3 & \quad \vert\ Ra & & \vee\text{E } 1, 2
\end{array}$$

This has shown that *if* we had $\neg Te$ as a premise, *then* we could prove Ra. In effect, we have proven $(\neg Te \rightarrow Ra)$. So, the conditional introduction rule (→I) will allow us to close the subproof and derive $(\neg Te \rightarrow Ra)$ in the main proof. Our final proof looks like this:

$$\begin{array}{r|lll}
1 & (Te \vee Ra) & & \\ \hline
2 & \quad \vert\ \underline{\neg Te} & & \\
3 & \quad \vert\ Ra & & \vee\text{E } 1, 2 \\
4 & (\neg Te \rightarrow Ra) & & \rightarrow\text{I } 2\text{–}3
\end{array}$$

Notice that the justification for applying the →I rule is the entire subproof. Usually, this will be more than two lines. The dash between the numbers indicates that we are citing the entire subproof as justification.

It may seem as if the ability to assume anything at all in a subproof would lead to chaos: Does it allow you to prove any conclusion from any premises? The answer is no, it does not. Consider this proof:

$$\begin{array}{r|lll}
1 & \mathcal{P} & & \\ \hline
2 & \quad \vert\ \underline{\mathcal{Q}} & & \\
3 & \quad \vert\ \mathcal{Q} & & \text{R } 2
\end{array}$$

It may seem as if this is a proof that you can derive any conclusions $\mathcal{Q}$ from any premise $\mathcal{P}$. When the vertical line for the subproof ends, the subproof is *closed*. In order to complete a proof, you must close all of the subproofs. And you cannot close the subproof and use the R rule again on line 4 to derive $\mathcal{Q}$ in the main proof. Once you close a subproof, you cannot refer back to individual lines inside it.

Closing a subproof is called 'discharging the assumptions of that subproof'. So, we can put the point this way: You cannot complete a proof until you have discharged all of the assumptions besides the original premises of the argument.

Of course, it is legitimate to do this:

1	$\mathcal{P}$	
2	$\quad \mathcal{Q}$	
3	$\quad \mathcal{Q}$	R 2
4	$(\mathcal{Q} \to \mathcal{Q})$	→I 2–3

This should not seem so strange, though. Since sentences of the form $(\mathcal{Q} \to \mathcal{Q})$ are tautologies, no particular premises are required to validly derive them. (Indeed, as we will see, tautologies follow from any or no premise.)

Put in a general form, the →I rule looks like this:

j	$\quad \mathcal{P}$	want $\mathcal{Q}$
k	$\quad \mathcal{Q}$	
l	$(\mathcal{P} \to \mathcal{Q})$	→I *j*–*k*

When we introduce a subproof, we typically write what we want to derive in the column. This is just so that we do not forget why we started the subproof if it goes on for five or ten lines. There is no 'want' rule. It is a note to ourselves and not formally part of the proof.

Although it is always permissible to open a subproof with any assumption you please, there is some strategy involved in picking a useful assumption. Start a subproof with an arbitrary, wacky assumption would just waste lines of the proof. In order to derive a conditional by the →I rule, for instance, you must assume that antecedent of the conditional in a subproof.

The →I rule also requires that the consequent of the conditional be the last line of the subproof. It is always permissible to close a subproof and discharge its assumptions, but it will not be helpful to do so until you get what you want.

Now that we have rules for the conditional, consider this argument:

$(Pa \to a = b)$

$(a = b \rightarrow Rab)$

$\therefore\ (Pa \rightarrow Rab)$

We begin the proof by writing the two premises as assumptions. Since the main logical operator in the conclusion is a conditional, we can expect to use the →I rule to derive it. For that we need to do a subproof—so we write in the antecedent of the conditional as assumption of a subproof:

1	$(Pa \rightarrow a = b)$	
2	$(a = b \rightarrow Rab)$	
3	│ Pa	want *Rab*

We made *Pa* available by assuming it in a subproof, allowing us to use →E on the first premise. This gives us $a = b$, which allows us to use →E on the second premise. Having derived *Rab*, we close the subproof. By assuming *Pa* we were able to prove *Rab*, so we apply the →I rule and finish the proof.

1	$(Pa \rightarrow a = b)$	
2	$(a = b \rightarrow Rab)$	
3	│ Pa	want *Rab*
4	│ $a = b$	→E 1, 3
5	│ Rab	→E 2, 4
6	$(Pa \rightarrow Rab)$	→I 3–5

6.2 Biconditional

The rules for the biconditional will be like double-barreled versions of the rules for the conditional.

In order to derive $(Te \leftrightarrow Wa)$, for instance, you must be able to prove *Wa* by assuming *Te and* prove *Te* by assuming *Wa*. The biconditional introduction rule (↔I) requires two subproofs. The subproofs can come in any order, and the second subproof does not need to come immediately after the first—but schematically, the rule works like this:

h	$\mid \mathcal{P}$	want $\mathcal{Q}$
i	$\mid \mathcal{Q}$	
j	$\mid \mathcal{Q}$	want $\mathcal{P}$
k	$\mid \mathcal{P}$	
l	$(\mathcal{P} \leftrightarrow \mathcal{Q})$	$\leftrightarrow$I h–i, j–k

The biconditional elimination rule ($\leftrightarrow$E) lets you do a bit more than the conditional elimination rule. If you have the left-hand subsentence of the biconditional, you can derive the right-hand subsentence. If you have the right-hand subsentence, you can derive the left-hand subsentence. This is the rule:

j	$(\mathcal{P} \leftrightarrow \mathcal{Q})$	
k	$\mathcal{P}$	
l	$\mathcal{Q}$	$\leftrightarrow$E j, k

j	$(\mathcal{P} \leftrightarrow \mathcal{Q})$	
k	$\mathcal{Q}$	
l	$\mathcal{P}$	$\leftrightarrow$E j, k

6.3 Negation

Here is a simple mathematical argument in English:

> Assume there is some greatest natural number. Call it A. That number plus one is also a natural number. Obviously, $A + 1 > A$. So there is a natural number greater than A. This is impossible, since A is assumed to be the greatest natural number. Therefore, there is no greatest natural number.

This argument form is traditionally called a '*reductio*'. Its full Latin name is '*reductio ad absurdum*', which means 'reduction to absurdity'. In a *reductio*, we assume something for the sake of argument—for example, that there is a greatest natural number. Then we show that the assumption leads to two contradictory sentences—for example, that A is the greatest natural number and that it is not. In this way, we show that the original assumption must have been false.

The basic rules for negation will allow for arguments like this. If we assume something and show that it leads to contradictory sentences, then we have proven the negation of the assumption. This is the negation introduction ($\neg$I) rule:

j	$\mathcal{P}$	for *reductio*
k	$\mathcal{Q}$	
$k+1$	$\neg\mathcal{Q}$	
$k+2$	$\neg\mathcal{P}$	$\neg$I j–$k+1$

For the rule to apply, the last two lines of the subproof must be an explicit contradiction: some sentence followed on the next line by its negation. We write 'for *reductio*' as a note to ourselves, a reminder of why we started the subproof. It is not formally part of the proof, and you can leave it out if you find it distracting.

To see how the rule works, suppose we want to prove an instance of the law of non-contradiction: $\neg(Ra \wedge \neg Ra)$. We can prove this without any premises by immediately starting a subproof. We want to apply $\neg$I to the subproof, so we assume $(Ra \wedge \neg Ra)$. We then get an explicit contradiction by $\wedge$E. The proof looks like this:

1		
2	$(Ra \wedge \neg Ra)$	for *reductio*
3	Ra	$\wedge$E 2
4	$\neg Ra$	$\wedge$E 2
5	$\neg(Ra \wedge \neg Ra)$	$\neg$I 2–4

The $\neg$E rule will work in much the same way. If we assume $\neg\mathcal{P}$ and show that it leads to a contradiction, we have effectively proven $\mathcal{P}$. So, the rule looks like this:

j	$\neg\mathcal{P}$	for *reductio*
k	$\mathcal{Q}$	
$k+1$	$\neg\mathcal{Q}$	
$k+2$	$\mathcal{P}$	$\neg$E j–$k+1$

6.4 Russian Doll proofs

We sometimes construct proofs in which one subproof appears inside another. We could call these 'Russian Doll proofs'.

Here is an example. You can establish using a truth table that the following inference is valid:

$$\big((Ra \land Teh) \to Sb\big) \qquad \therefore \qquad \big(Ra \to (Teh \to Sb)\big)$$

To prove this, we begin as usual by writing down our premise:

$$\begin{array}{ll}
1 & \vert\ \underline{\big((Ra \land Teh) \to Sb\big)}
\end{array}$$

Now our conclusion is a conditional, so it's a good bet that we'll prove it using →I. So, our proof will in the end look something like this:

$$\begin{array}{lll}
1 & \vert\ \underline{\big((Ra \land Teh) \to Sb\big)} & \\
2 & \vert\ \quad \vert\ \underline{Ra} & \\
 & \vert\ \quad \vert\ \dots & \\
? & \vert\ \quad \vert\ (Teh \to Sb) & \\
 & \vert\ \big(Ra \to (Teh \to Sb)\big) & \to\text{I } 2\text{–}?
\end{array}$$

To complete the proof, we need to fill in the blank with a proof of $(Teh \to Sb)$. How will we prove this? Well, this is another conditional, so it's likely that we'll want to use →I again. So, our proof will look like this:

$$\begin{array}{lll}
1 & \vert\ \underline{\big((Ra \land Teh) \to Sb\big)} & \\
2 & \vert\ \quad \vert\ \underline{Ra} & \\
3 & \vert\ \quad \vert\ \quad \vert\ \underline{Teh} & \\
 & \vert\ \quad \vert\ \quad \vert\ \dots & \\
?? & \vert\ \quad \vert\ \quad \vert\ Sb & \\
? & \vert\ \quad \vert\ (Teh \to Sb) & \to\text{I } 3\text{–}??\\
 & \vert\ \big(Ra \to (Teh \to Sb)\big) & \to\text{I } 2\text{–}?
\end{array}$$

To complete the thing, all we need do is fill in the blank—which turns out not to be so hard:

1	| $((Ra \land Teh) \to Sb)$	
2	| | Ra	
3	| | | Teh	
4	| | | $(Ra \land Teh)$	∧I 2, 3
5	| | | Sb	→E 1, 4
6	| | $(Teh \to Sb)$	→I 3–5
7	| $(Ra \to (Teh \to Sb))$	→I 2–6

6.5 Proving tautologies and tautological equivalences

We've seen in chapter three that we can use truth tables to determine whether a sentence is a tautology or whether two or more sentences are tautologically equivalent. Now that we've got a truth-functional system of natural deduction, we are able to establish these things in our proof system.

We defined a tautology as a sentence whose completed truth table receives all Ts under the sentence's main connective. This means that a tautology is a sentence that is necessarily true simply in virtue of the meanings of the truth-functional connectives. Since the rules for our proof system are also grounded in the meanings of the connectives, we should be able to derive tautologies from any set of premises. Indeed, we can prove tautologies from *no* premises. Consider the tautology $(Pa \lor \neg Pa)$. The following proof shows how to derive the sentence from no premise:

1	|	
2	| | $\neg(Pa \lor \neg Pa)$	
3	| | | $\neg Pa$	
4	| | | $(Pa \lor \neg Pa)$	∨I 3
5	| | | $\neg(Pa \lor \neg Pa)$	R 2
6	| | Pa	¬E 3–5
7	| | $(Pa \lor \neg Pa)$	∨I 6
8	| | $\neg(Pa \lor \neg Pa)$	R 2
9	| $(Pa \lor \neg Pa)$	¬E 2–8

Similarly, we saw in chapter three that two or more sentences are tautologically equivalent just in case they share the same truth value in every row of their joint truth table. Consider the following sentences:

1. $\neg(Ra \wedge Gb)$
2. $(\neg Ra \vee \neg Gb)$

A joint truth table will reveal that 1 and 2 are tautologically equivalent. Since they are tautologically equivalent, every row that assigns a T to 1 also assigns a T 2. So, 2 is a tautological consequence of 1. Furthermore, every row that assigns a T to 2 also assigns a T to 1. So, 1 is a tautological consequence of 2. Our proof system will allow us to derive 1 from 2 and vice versa, thus showing that they are tautologically equivalent. Indeed, since 1 and 2 are tautologically equivalent, we can derive the sentence $\big(\neg(\text{Ra} \wedge \text{Gb}) \leftrightarrow (\neg\text{Ra} \vee \neg\text{Gb})\big)$ from no premises. We will leave this for practice exercise G.

Summary of derivation rules in TFL covered in this chapter

Conditional Introduction ($\to$I)

$$\begin{array}{r|l|l}
j & & \mathcal{P} \\ \cline{3-3}
k & & \mathcal{Q} \\
l & \multicolumn{2}{l}{(\mathcal{P} \to \mathcal{Q}) \qquad \to\text{I } j\text{–}k}
\end{array}$$

Biconditional Introduction ($\leftrightarrow$I)

$$\begin{array}{r|l|l}
h & & \mathcal{P} \\ \cline{3-3}
i & & \mathcal{Q} \\
j & & \mathcal{Q} \\ \cline{3-3}
k & & \mathcal{P} \\
l & \multicolumn{2}{l}{(\mathcal{P} \leftrightarrow \mathcal{Q}) \qquad \leftrightarrow\text{I } h\text{–}i,\ j\text{–}k}
\end{array}$$

Biconditional Elimination (↔E)

$$\begin{array}{l|ll} j & (\mathcal{P} \leftrightarrow \mathcal{Q}) & \\ k & \mathcal{P} & \\ l & \mathcal{Q} & \leftrightarrow\text{E } j, k \end{array} \qquad \begin{array}{l|ll} j & (\mathcal{P} \leftrightarrow \mathcal{Q}) & \\ k & \mathcal{Q} & \\ l & \mathcal{P} & \leftrightarrow\text{E } j, k \end{array}$$

Negation Introduction (¬I)

$$\begin{array}{l|l|ll} j & & \underline{\mathcal{P}} & \text{for } \textit{reductio} \\ k & & \mathcal{Q} & \\ k+1 & & \neg\mathcal{Q} & \\ k+2 & \neg\mathcal{P} & & \neg\text{I } j\text{–}k+1 \end{array}$$

Negation Elimination (¬E)

$$\begin{array}{l|l|ll} j & & \underline{\neg\mathcal{P}} & \text{for } \textit{reductio} \\ k & & \mathcal{Q} & \\ k+1 & & \neg\mathcal{Q} & \\ k+2 & \mathcal{P} & & \neg\text{E } j\text{–}k+1 \end{array}$$

Practice exercises

Part A * Identify the mistake in each of the following "proofs".

1.

$$
\begin{array}{rlll}
1 & \multicolumn{2}{|l}{\underline{(Rab \lor Map)}} & \\
2 & \multicolumn{1}{|l}{} & \multicolumn{1}{|l}{\underline{\neg Rab}} & \\
3 & \multicolumn{1}{|l}{} & \multicolumn{1}{|l}{Map} & \lor\text{E } 1, 2 \\
4 & \multicolumn{1}{|l}{} & \multicolumn{1}{|l}{(\neg Rab \to Map)} & \to\text{I } 2\text{–}3
\end{array}
$$

2.

$$
\begin{array}{rlll}
1 & \multicolumn{2}{|l}{\underline{\neg Rt}} & \\
2 & \multicolumn{1}{|l}{} & \multicolumn{1}{|l}{\underline{(Rt \lor p = q)}} & \\
3 & \multicolumn{1}{|l}{} & \multicolumn{1}{|l}{p = q} & \lor\text{E } 1, 2 \\
4 & \multicolumn{2}{|l}{\big((Rt \lor p = q) \to p = q\big)} & \to\text{I } 2, 3
\end{array}
$$

3.

$$
\begin{array}{rlll}
1 & \multicolumn{2}{|l}{\big[Ap \to \big((Bs \land Emp) \lor \neg Cn\big)\big]} & \\
2 & \multicolumn{2}{|l}{Ap} & \\
3 & \multicolumn{2}{|l}{\underline{\neg\neg Cn}} & \\
4 & \multicolumn{1}{|l}{} & \multicolumn{1}{|l}{\underline{Dr}} & \\
5 & \multicolumn{1}{|l}{} & \multicolumn{1}{|l}{\big((Bs \land Emp) \lor \neg Cn\big)} & \to\text{E } 1, 2 \\
6 & \multicolumn{1}{|l}{} & \multicolumn{1}{|l}{(Bs \land Emp)} & \lor\text{E } 3, 5 \\
7 & \multicolumn{1}{|l}{} & \multicolumn{1}{|l}{Bs} & \land\text{E } 6 \\
8 & \multicolumn{1}{|l}{} & \multicolumn{1}{|l}{Emp} & \land\text{E } 6 \\
9 & \multicolumn{2}{|l}{(Dr \to Emp)} & \to\text{I } 4\text{–}8 \\
10 & \multicolumn{2}{|l}{\big((Dr \to Emp) \land Bs\big)} & \land\text{I } 7, 9
\end{array}
$$

---

[*] Many thanks to Kesavan Thanagopal for offering these exercises, originally created for his PHIL110 *Introduction to Logic and Reasoning* tutorials.

4.

1	$\big(Pab \to (Qee \land Rst)\big)$	
2	$\quad\vert\ Pab$	
3	$\quad\vert\ (Qee \land Rst)$	$\to$E 1, 2
4	$\quad\vert\ Qee$	$\land$E 3
5	$\quad\vert\ Rst$	$\land$E 3
6	$(Pab \to Qee)$	$\to$I 2–4
7	$(Pab \to Rst)$	$\to$I 2–5
8	$\big((Pab \to Qee) \land (Pab \to Rst)\big)$	$\land$I 6, 7

Part B Consider the following inferences. In each case, establish that the inference is valid by providing a natural deduction proof.

1. $\big(Tu \to (Rc \land Oae)\big)$
   $\therefore (Tu \to Rc)$

2. $(Tu \to Rc)$
   $\therefore \big(Tu \to (Rc \lor a = a)\big)$

3. Gb
   $\therefore \big(a = b \to (Gb \land a = b)\big)$

Part C Consider the following inferences. In each case, establish that the inference is valid by providing a natural deduction proof.

1. $\neg Tu$
   $\therefore \neg(Tu \land a = b)$

2. $\neg(Ugg \lor Ta)$
   $\therefore \neg Ugg$

3. $\neg Ta$
   $(a = b \to Ta)$
   $\therefore a \neq b$

Part D Consider the arguments in bold below. In each case, symbolize the argument, and establish whether it's valid, explaining your answer in detail.

1. At the Midsummer's Fair in 16th century Paris, cat-burning was a regular attraction. A large net containing many cats would be slowly lowered over a bonfire. Spectators would delight in watching the animals screech as they slowly burnt to death. Today, we all recognize that this is barbaric. However, many people fail to understand that similarly cruel practices are common in the meat industry today. For example, in the USA every year tens of millions of piglets are castrated, without pain relief. **If the Parisian cat-burnings were immoral, as they surely were, modern pig farming is immoral. And if modern pig farming is immoral, it is surely immoral to buy pork. So, it is immoral to buy pork.**

2. Research in history has established that moral rules have changed over the centuries. For example, the institution of slavery was once widely accepted, but now slavery is forbidden and deplored everywhere. Moreover, research in anthropology has established that moral rules vary across the world today. For example, in some places it is considered okay to drink alcohol, but in other places it is forbidden. This has important implications for our understanding of the nature of morality. **If moral rules were created by God, then they would be the same in all times and places. But as we have seen, moral rules are not the same in all times and places. Now, clearly, either moral rules were created by God or they are a social construction. So, moral rules are a social construction.**

3. **If my passport is on the desk in my office, I'll miss my flight. And, you know, my passport is on the desk in my office, unless of course I left it at home. But if my passport were at home, my wife would have found it, and she hasn't. So, I'm going to miss my flight.**

Part E Consider the following two arguments. One is valid; the other is invalid.

$$Ra, \quad (Ra \rightarrow Tu) \quad \therefore \quad \big((Ra \lor Gc) \land Tu\big)$$
$$Ra, \quad (Ra \rightarrow Tu) \quad \therefore \quad \big((Ra \land Gc) \land Tu\big)$$

Give a proof to establish the validity of one argument. Use a truth table to establish the invalidity of the other.

Part F Consider the following inferences. In each case, establish that the inference is valid by providing a natural deduction proof.

1. $(Ra \rightarrow (b = c \wedge c = d)) \quad \therefore \quad \big((Ra \rightarrow b = c) \wedge (Ra \rightarrow c = d)\big)$

2. $\neg(Ra \vee Gb) \quad \therefore \quad (\neg Ra \wedge \neg Gb)$

3. $(\neg Ra \wedge \neg Gb) \quad \therefore \quad \neg(Ra \vee Gb)$

Part G Using a natural deduction proof, show that the following statement can be derived using no premises at all. (*challenging*)

$$\big(\neg(Ra \wedge Gb) \leftrightarrow (\neg Ra \vee \neg Gb)\big)$$

Part II

First-order logic

Chapter 7

Introducing the quantifiers

7.1 Introduction

In this chapter, we introduce a logical language called FOL, which is an extension of TFL. It is a version of *first-order logic*, which along with the truth-functional connectives and the identity symbol, includes quantifiers like *all* and *some*. Because of this, it is sometime called 'quantified' or 'predicate logic'. 'First-order' refers to the kind of things over which we can quantify. In FOL, we quantify over individual objects or things. FOL affords us with a gain in expressive power; we can thus prove more things in FOL than we could prove in TFL.

Suppose we are talking about everyone in our logic class. Now, consider the following argument, which is valid in English:

> If everyone knows logic, then either no one will be confused or everyone will. Everyone will be confused only if we try to believe a contradiction. Everyone in our logic class knows logic. Therefore, if we don't try to believe a contradiction, then no one will be confused.

We quickly run into problems when we try to translate this argument in TFL. It is easy enough to start our symbolization key with a list of predicates:

Lx: x knows logic
Cx: x is confused
Bx: x tries to believe a contradiction

One initial problem is that the argument doesn't contain any singular term. It might be tempting to simply treat 'everyone' and 'no one' as singular terms and add the following to our symbolization key:

$\boldsymbol{e}$: everyone
$\boldsymbol{n}$: no one

The translation would be something like this:

$$\begin{array}{rl} & \big(Le \to (Cn \vee Ce)\big) \\ & (Ce \to Be) \\ & Le \\ \therefore & (\neg Be \to Cn) \end{array}$$

As it turns out, this is a valid argument in TFL. (You can either do a joint truth table to check for this or construct in a proof in our proof system.) But we cannot treat 'everyone' and 'no one' as singular terms since neither expression refers to a specific individual. So while the argument is valid in TFL, it is not a correct translation of the argument given in English above.

That we cannot treat quantified phrases as singular terms becomes even clearer when we consider another argument that is valid in English:

> Willard is a logician. All logicians wear funny hats. So, Willard wears a funny hat.

Now consider the following symbolization key:

$\boldsymbol{Lx}$: x is a logician
$\boldsymbol{Fx}$: x wears a funny hat
$\boldsymbol{i}$: Willard
$\boldsymbol{a}$: all

One might be tempted to treat 'all' as a singular term and translate the argument in the following way:

$$\begin{array}{rl} & Li \\ & (La \to Fa) \\ \therefore & Fi \end{array}$$

The problem is that this argument is clearly *invalid* in TFL. So, something has gone very wrong. We started with a valid argument in English and ended up with an invalid argument in TFL.

The problem is that the symbolization in TFL leaves out a very important aspect of the argument's structure: it overlooks *quantifier structure*. The sentence 'All logicians wear funny hats' is about both logicians and hat-wearing; it is not about some individual named 'all'. By not translating this structure, we lose the connection between Willard's being a logician and Willard's wearing a hat. This tells us that we cannot treat quantifiers as singular terms; we need to find a way to capture their logical structure.

The mistake we've made while symbolizing both arguments is a natural one. Atomic sentences in TFL are composed of *n*-ary predicates and *n* terms, where the terms in question are individual constants. In translating English sentences into TFL, we are forced to look for constants even when there are none. But while we've made mistakes, these are nevertheless the best symbolizations we can give for these arguments *in* TFL. So, clearly, we need something better.

In order to symbolize arguments that rely on quantifier structure, we need to develop a different logical language. We will call this language 'first-order logic' (FOL). FOL includes TFL, but also contains a *universal* and an *existential quantifier*. We introduce these two quantifiers in the next section.

7.2 The quantifiers

Suppose we have the following symbolization key:

Ax: x is angry
Hx: x is happy
Txy: x is taller than y
d: Donald
g: Gregor
m: Marybeth

Now consider the following sentences:

1. Everyone is happy.
2. Everyone loves Donald.
3. Someone is angry.

It might be tempting to translate sentence 1 as $\big((\mathrm{Hd} \wedge \mathrm{Hg}) \wedge \mathrm{Hm}\big)$. Yet, this would only say that Donald, Gregor, and Marybeth are happy. We want to say that *everyone* is happy—including people whose names we don't know. In order to do this, we introduce the '∀' symbol. This is called the UNIVERSAL QUANTIFIER.

The universal quantifier must always be followed by a variable and a formula (we will give a precise definition of a formula and a sentence in FOL later in the chapter). We can translate sentence 1 as $\forall x\, Hx$. Paraphrased in English, it means 'For all x, x is happy'.

Sentence 2 can be paraphrased as 'For all x, x loves Donald'. This translates as $\forall x\, Lxd$.

In the quantified sentences, the variable x is serving as a kind of placeholder. The expression $\forall x$ means that you can pick out anyone and put them in as x. There is no special reason to use x rather than some other variable. The sentence $\forall x\, Hx$ means exactly the same thing as $\forall y\, Hy$, $\forall z\, Hz$, and $\forall x_5\, Hx_5$.

To translate sentence 3, we introduce another new symbol: the EXISTENTIAL QUANTIFIER '$\exists$', which roughly corresponds to the English words 'something' or 'at least one thing'. Like the universal quantifier, the existential quantifier requires a variable. Sentence 3 can be translated as $\exists x\, Ax$. This can be paraphrased in English as 'for some x, x is angry'. More precisely, it means that there is at least one angry person.

Consider these further sentences:

4. No one is angry.
5. There is someone who is not happy.
6. Not everyone is angry.

Sentence 4 can be paraphrased as 'It is not the case that someone is angry'. This can be translated using negation and an existential quantifier: $\neg\exists x\, Ax$. We could also interpret sentence 4 as 'Everyone is not angry'. With this in mind, it can be translated using negation and a universal quantifier: $\forall x\, \neg Ax$. More generally, and as we'll in chapter 9, sentences of forms $\neg\exists x\, Ax$ and $\forall x\, \neg Ax$ are logically equivalent.

Sentence 5 is most naturally paraphrased as 'There is some x such that x is not happy.' This becomes $\exists x\, \neg Hx$. Equivalently, we could write $\neg\forall x\, Hx$.

Sentence 6 is most naturally translated as $\neg\forall x\, Ax$. Again, as we will see later on, this is logically equivalent to $\exists x\, \neg Ax$.

7.3 Universe of discourse

Given the symbolization key we've been using, $\forall x\, Hx$ means 'Everyone is happy.' Who is included in this *everyone*? When we use sentences like this in English, we usually do not mean everyone alive on the Earth. We certainly don't mean everyone who was ever alive and who will ever live. We mean something more modest: everyone in the building, everyone in the class, or everyone in the room.

In order to eliminate this ambiguity, we will need to specify a UNIVERSE OF DISCOURSE—abbreviated 'UD'. The UD is the set of things that we are talking about and over which we can quantify. So, if we want to talk about people in Chicago, we define the UD to be people in Chicago. We write this at the beginning of the symbolization key, like this:

UD: People in Chicago

The quantifiers *range over* the universe of discourse. Given this UD, $\forall x$ means 'Everyone in Chicago' and $\exists x$ means 'Someone in Chicago'. Each constant names some member of the UD, so we can only use this UD with the symbolization key above if Donald, Gregor, and Marybeth are all in Chicago. If we want to talk about people in places besides Chicago, then we need a different UD that includes them.

In FOL, the UD must be *non-empty*; that is, it must include at least one thing. It is possible to construct formal languages that allow for empty UDs, but this introduces complications.

Even allowing for a UD with just one member can produce some strange results. Suppose we have this as a symbolization key:

UD: The Eiffel Tower
Px: x is in Paris

The sentence $\forall x\, Px$ might be paraphrased in English as 'Everything is in Paris.' Yet, that would be misleading. It means that everything in the UD is in Paris. This UD contains only the Eiffel Tower, so with this symbolization key, $\forall x\, Px$ just means that the Eiffel Tower is in Paris.

7.4 Translating to FOL

We now have all of the pieces of FOL. Translating more complicated sentences will only be a matter of knowing the right way to combine predicates, constants, quantifiers, variables, and connectives. Consider these sentences:

7. Every coin in my pocket is a quarter.
8. Some coin on the table is a dime.
9. Not all the coins on the table are dimes.
10. None of the coins in my pocket are dimes.

In providing a symbolization key, we need to specify a UD. Since we are talking about coins in my pocket and on the table, the UD must at least contain all of those coins. Since we are not talking about anything besides coins, we let the UD be all coins. Since we are not talking about any specific coins, we do not need to define any constant. So, we use this key:

UD: All coins
***Px*:** x is in my pocket
***Tx*:** x is on the table
***Qx*:** x is a quarter
***Dx*:** x is a dime

Sentence 7 is most naturally translated with a universal quantifier. The universal quantifier says something about everything in the UD, not just about the coins in my pocket. Sentence 7 means that, for any coin, if that coin is in my pocket, then it is a quarter. So, we can translate $\forall x\,(Px \rightarrow Qx)$.

Since sentence 7 is about coins that are both in my pocket *and* that are quarters, it might be tempting to translate it using a conjunction. However, the sentence $\forall x\,(Px \wedge Qx)$ would mean that everything in the UD is both in my pocket and a quarter: All the coins that exist are quarters in my pocket.

This would be a crazy thing to say, and would mean something very different than sentence 7.

Sentence 8 is most naturally translated with an existential quantifier. It says that there is some coin which is both on the table and which is a dime. So, we can translate it as $\exists x\,(Tx \wedge Dx)$.

Notice that we needed to use a conditional with the universal quantifier, but we used a conjunction with the existential quantifier. What would it mean to write $\exists x\,(Tx \rightarrow Dx)$? Probably not what you think. It means that there is some member of the UD which would satisfy the subformula; roughly speaking, there is some a such that $(Ta \rightarrow Da)$ is true. In TFL, $(Ta \rightarrow Da)$ is logically equivalent to $(\neg Ta \vee Da)$, and this will also hold in FOL. So $\exists x\,(Tx \rightarrow Dx)$ is true if there is some a such that $(\neg Ta \vee Da)$; i.e., it is true if some coin is either not on the table or is a dime. Of course, there is a coin that is not on the table—there are coins in lots of other places. So $\exists x\,(Tx \rightarrow Dx)$ is trivially true. A conditional will usually be the natural connective to use with a universal quantifier, but a conditional within the scope of an existential quantifier can do very strange things. As a general rule, do not put conditionals in the scope of existential quantifiers unless you are sure that you need one.

Sentence 9 can be paraphrased as, 'It is not the case that every coin on the table is a dime.' So, we can translate it as $\neg\exists x\,(Tx \wedge Dx)$. You might look at sentence 9 and paraphrase it instead as, 'Some coin on the table is not a dime.' You would then translate it as $\exists x\,(Tx \wedge \neg Dx)$. Although it is probably not obvious, these two translations are logically equivalent.

Sentence 10 can be paraphrased as, 'It is not the case that there is some dime in my pocket.' This can be translated as $\neg\exists x\,(Px \wedge Dx)$. It might also be paraphrased as 'Everything in my pocket is a non-dime' and then could be translated as $\forall x\,(Px \rightarrow \neg Dx)$. Again, the two translations are logically equivalent. Both are correct translations of sentence 10.

We can now translate the arguments from pp. 87-8, the ones that motivated the need for quantifiers:

> Willard is a logician. All logicians wear funny hats. So, Willard wears a funny hat.

Let's work with the following symbolization key:

UD: People
***Lx*:** *x* is a logician
***Fx*:** *x* wears a funny hat
***w*:** Willard

Translating, we get:

$$\begin{array}{rl} & Lw \\ & \forall x\,(Lx \rightarrow Fx) \\ \therefore & Fw \end{array}$$

The other argument was:

> If everyone knows logic, then either no one will be confused or everyone will. Everyone will be confused only if we try to believe a contradiction. This is a logic class, so everyone knows logic. Therefore, if we don't try to believe a contradiction, then no one will be confused.

Here is a symbolization key:

UD: People in this class
***Lx*:** *x* knows logic
***Cx*:** *x* is confused
***Bx*:** *x* tries to believe a contradiction

Translating, we get:

$$\begin{array}{rl} & \big(\forall x\,Lx \rightarrow (\neg\exists y\,Cy \vee \forall z\,Cz)\big) \\ & (\forall x\,Cx \rightarrow \forall y\,By) \\ & \forall x\,Lx \\ \therefore & (\forall x\,\neg Bx \rightarrow \neg\exists y\,Cy) \end{array}$$

Notice that we've used different variables in the antecedents and consequents of the first two premises and the conclusion. Strictly speaking, we didn't need to do this, but it is good practice: it is potentially confusing to give one variable two different jobs in one statement.

Both arguments in FOL capture the structure that was left out of the TFL translations, and both arguments are valid in FOL.

7.5 Picking a UD

We said earlier that if you are only talking about a specific subset of things in the world (e.g., coins), a natural choice for a UD is simply the set of all coins. But we could have made a different choice. We could have let the UD be the set of all things.

UD: Everything

This is particularly useful if you want to talk about a wider range of disparate things. Even with such a UD, we can translate sentences 7 – 10 into FOL. In order to do that, we would simply need to add to our symbolization key a symbol that allows us to talk about the property of being a coin:

Cx: x is a coin
Px: x is in my pocket
Tx: x is on the table
Qx: x is a quarter
Dx: x is a dime

A paraphrase of sentence 7 would be 'For any object x, if x is a coin and x is in my pocket, then x is a quarter'. The FOL translation would be $\forall x\big((Cx \land Px) \to Qx\big)$. Sentences 8 – 10 could be translated thusly:

11. $\exists x\big((Cx \land Tx) \land Dx\big)$
12. $\neg\forall x\big((Cx \land Tx) \to Dx\big)$
13. $\neg\exists x\big((Cx \land Tx) \land Dx\big)$

7.6 Sentences of FOL

In this section we provide a formal definition of a *sentence* of FOL. We first begin by giving a recursive definition of a *well-formed formula* (wff).

There are six kinds of symbols in FOL:

Predicates with subscripts, as needed	$A, B, C_2, \ldots, Z, =$
Constants with subscripts, as needed	$a, b, c, \ldots, w, 1, 2, \ldots$
Variables with subscripts, as needed	$x, y, z, x_1, y_1, z_1, x_2, \ldots$
Connectives	$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
Parentheses and brackets	(,), [,]
Quantifiers	$\forall, \exists$

By definition, a TERM of FOL is either a constant or a variable.

An ATOMIC FORMULA of FOL is an n-ary predicate followed by n terms.

With this in mind, we can move on to give a recursive definition of a wff:

1. Every atomic formula is a wff.
2. If $\mathcal{A}$ is a wff, then $\neg\mathcal{A}$ is a wff.
3. If $\mathcal{A}$ and $\mathcal{B}$ are wffs, then $(\mathcal{A} \wedge \mathcal{B})$ is a wff.
4. If $\mathcal{A}$ and $\mathcal{B}$ are wffs, then $(\mathcal{A} \vee \mathcal{B})$ is a wff.
5. If $\mathcal{A}$ and $\mathcal{B}$ are wffs, then $(\mathcal{A} \rightarrow \mathcal{B})$ is a wff.
6. If $\mathcal{A}$ and $\mathcal{B}$ are wffs, then $(\mathcal{A} \leftrightarrow \mathcal{B})$ is a wff.
7. If $\mathcal{A}$ is a wff, χ is a variable, and $\mathcal{A}$ contains no quantified wff with χ already bound by a quantifier, then $\forall\chi\,\mathcal{A}$ is a wff.
8. If $\mathcal{A}$ is a wff, χ is a variable, and $\mathcal{A}$ contains no quantified wff with χ already bound by a quantifier, then $\exists\chi\,\mathcal{A}$ is a wff.
9. All and only wffs of FOL can be generated by applications of these rules.

Notice that 'χ' and 'y' that appear in the definition above are not the variables 'x' and 'y'. They are meta-variables that stand for any variable of FOL. So, $\forall x\, Ax$ is a wff, but so are $\forall y\, Ay$, $\forall x_4\, Ax_4$, and $\forall z_9\, Az_9$.

Now, that we've got a definition of a wff, we can move on to give a definition of a sentence of FOL. First, note that a sentence is something can be either true or false. In TFL, every wff was a sentence. This will not be the case in FOL. Consider the following symbolization key:

UD: People
***Lxy*:** x loves y
***b*:** Boris

Consider the expression Lzz. It is an atomic formula: a binary predicate followed by two terms. All atomic formulas are wffs, so Lzz is a wff. Does it mean anything? You might think that it means that z loves himself in the same way that Lbb means that Boris loves himself. Yet, z is a variable; it does not name some person that way a constant would. The wff does not tell us how to interpret z. Does it mean everyone? Anyone? Someone? If we had a quantifier in front, it would tell us how to interpret z. For instance, $\exists z\, Lzz$ would mean that someone loves themselves.

In order to make sense of a variable, we need a quantifier to tell us how to interpret that variable. The SCOPE of a quantifier is the part of the formula where the quantifier tells us to interpret the variable; it is the subformula for which the quantifier is the main logical operator.

In order to be precise about this, we define a BOUND VARIABLE to be an occurrence of a variable x that is within the scope of a quantifier. A FREE VARIABLE is an occurrence of a variable that is not bound.

For example, consider the wff $\big(\forall x\, (Rx \vee Dy) \rightarrow \exists z\, (Ex \rightarrow Lzx)\big)$. The scope of the universal quantifier is $(Rx \vee Dy)$, so the first x is bound by the universal quantifier, but the second and third xs are free. No quantifier binds the variable y, so y is free. The scope of the existential quantifier is $(Ex \rightarrow Lzx)$, so the occurrence of z is bound by it.

We define a SENTENCE of FOL as a wff of FOL that contains no free variables. The wff above is not a sentence. It is a mere wff.

7.7 Satisfaction

The sentence Pa is true just in case whatever object named by the constant 'a' has the property predicated by 'P'. With quantifiers, things are different. For example, we cannot say that that the sentence $\forall x\, Px$ is true just in case x is a P. Again, 'x' is not a constant but a variable; it doesn't refer to any object. For the same reason, we cannot say that $\forall x\, Px$ is true just in case all x's are P's. There is no objected named 'x'.

In order to talk about the conditions under which quantified sentences are true, we appeal to the concept of *satisfaction*. We say that a universal quantified sentence of the form $\forall x\, \mathcal{A}$ is true just in case *every* object in the UD satisfies the wff within the scope of $\forall x$. For example, the sentence $\forall x\, Px$ is true just in case every object in the UD satisfies Px. In turn, every object in the UD satisfies Px just in case every object in the UD is a P.

Similarly, we say that an existentially quantified sentence of the form $\exists x\, Px$ is true just in case *at least one* object in the UD satisfies the wff within the scope of $\exists x$. For example, the sentence $\exists x\, Px$ is true just in case there is at least one object in the UD that satisfies Px. And to say that at least one object in the UD satisfies Px is just to say that at least one object in the UD is a P.

The truth conditions for quantified sentences can lead us to counterintuitive results. For example, suppose we're working with the following symbolization key:

UD: Animals
Mx: x is a monkey
Sx: x knows sign language
Rx: x is a refrigerator

Now consider the following three sentences:

14. Every monkey knows sign language.
15. Some monkey knows sign language.
16. Every refrigerator knows sign language.

Their FOL translations are:

17. $\forall x\,(Mx \rightarrow Sx)$
18. $\exists x\,(Mx \wedge Sx)$
19. $\forall x\,(Rx \rightarrow Sx)$

One might think that 15 follows from 14. That is, it might be tempting to think that if 14 is true, then 15 must be true. Furthermore, one might think that sentence 16 is obviously false. But while both of these thoughts are intuitive, they are mistaken. So, let us end this chapter by explaining why 15 *does not* follow from 14, and why 16 is *true* in this context.

The first thing to note is that the predicate 'R' does not apply to anything in the UD: no animal is a refrigerator. But that is not a problem. A predicate need not apply to anything in the UD. A predicate that applies to nothing in the UD is called an 'empty predicate'.

Second, it is tempting to say that sentence 14 entails sentence 15; that is, if every monkey knows sign language, then it must be that some monkey knows sign language. This is a valid inference in Aristotelian logic: All Ms are S, $\therefore$ Some M is S. However, the entailment does not hold in FOL. It is possible for the sentence $\forall x\,(Mx \rightarrow Sx)$ to be true even though the sentence $\exists x\,(Mx \wedge Sx)$ is false. How can this be?

We have defined $\forall$ and $\exists$ in such a way that $\forall x \neg \mathcal{A}$ is equivalent to $\neg \exists x\, \mathcal{A}$. As such, the universal quantifier doesn't involve the existence of anything—only non-existence. If sentence 18 were true, then there are no monkeys who don't know sign language. If there were no monkeys, then it would trivially follow that there were no monkeys who don't know sign language. So, if there were no monkeys, then $\forall x\, (Mx \rightarrow Sx)$ would be true and $\exists x\, (Mx \wedge Sx)$ would be false.

Similar considerations explain why the sentence 19 is true in the context above. The UD specified above is the set of animals. Since there is no refrigerator in the UD, the predicate R is an empty predicate. The sentence $\forall x\, (Rx \rightarrow Sx)$ can be interpreted to mean that there is no refrigerator that doesn't know sign language. But since there is no refrigerator in the UD, it trivially follows that there is no refrigerator that doesn't know sign language.

The notion of *satisfaction* helps make further sense of this. The sentence $\forall x\, (Rx \rightarrow Sx)$ states that every object in the domain satisfies $(Rx \rightarrow Sx)$. But since $(Rx \rightarrow Sx)$ is logically equivalent to $(\neg Rx \vee Sx)$, the sentence $\forall x\, (Rx \rightarrow Sx)$ states that every object in the UD is either not a refrigerator or knows sign language. Since the UD is the set of animals (and animals are not refrigerators), it follows that every object in the UD satisfies $(Rx \rightarrow Sx)$. That is why the sentence $\forall x\, (Rx \rightarrow Sx)$ is true. We can say of such sentences that they are *vacuously true*.

Summary of logical notions

- The UNIVERSAL QUANTIFIER ($\forall$) stands for 'all' and 'every'.
- The EXISTENTIAL QUANTIFIER ($\exists$) stands for 'some', 'at least one', 'there exists at least one'.
- The SCOPE of a quantifier is the subformula for which the quantifier is the main logical operator.
- A TERM of FOL is either a constant or a variable.
- An ATOMIC FORMULA is any *n*-ary predicate followed by *n* terms.
- A BOUND VARIABLE is a variable that is within the scope of a quantifier.
- A FREE VARIABLE is a variable that is not bound.
- A SENTENCE of FOL as a wff of FOL that contains no free variables.

Practice exercises

Part A Using the key given below, translate the numbered sentences.

UD: The people at a certain party
***Mx*:** x is a mathematician
***Px*:** x is a philosopher
***Dxy*:** x admires y
***s*:** Ashni
***b*:** Ben

1. Ashni is a mathematician.
2. Ashni is a philosopher.
3. Ashni is either a mathematician or a philosopher.
4. Ashni admires Ben.
5. Ben admires Ashni.
6. Ashni and Ben admire each other.
7. Ashni is a mathematician and she admires Ben.
8. Ashni and Ben are mathematicians who admire each other.
9. Everyone is a mathematician. (i.e. Everyone at the party is a mathematician.)
10. Everyone is either a mathematician or a philosopher.
11. Everyone admires Ashni.
12. Ashni admires everyone.
13. Every mathematician admires Ashni.
14. Everyone who admires Ashni is either a mathematician or a philosopher.

Part B Devise your own key, and then translate the numbered sentences.

1. Jerry is a mouse.
2. Jerry is a mammal.
3. Jumbo is an elephant.
4. Jumbo is bigger than Jerry.
5. Every mouse is a mammal / Mice are mammals / A mouse is always a mammal.
6. Mice and elephants are mammals.

Part C Which of the following inferences are valid?

1.

Everyone at the party is wearing red.
Everyone who is wearing red is cool.
∴ Everyone at the party is cool.

2.

Nobody at the party is wearing red.
Nobody who is wearing red is cool.
∴ Nobody at the party is cool.

3.

Everyone who likes Nickelback is cool.
∴ Everyone who's cool likes Nickelback.

4.

Everyone who likes Nickelback is cool.
∴ Anyone who isn't cool doesn't like Nickelback.

5.

Nobody who likes Nickelback is cool.
∴ Nobody who's cool likes Nickelback.

Part D Using the key given below, fill in the gaps in the following table:

UD: The people at a certain party
***Cx*:** x is chatting
***Dx*:** x is dancing
***Lxy*:** x loves y
***a*:** Ashni
***b*:** Ben

English	**Symbols**
Someone loves Ashni.	$\exists x\, Lxa$
Ben and someone love each other.	
	$\exists x \big((Dx \land Cx) \land Lxb\big)$
	$(\exists x\, Dx \land \exists y\, \neg Dy)$
If someone is chatting, then someone is dancing.	
Either everyone is dancing, or someone is not dancing.	

Part E Using the symbolization key given, translate each English-language sentence into FOL.

UD: All animals
Ax: x is an alligator
Mx: x is a monkey
Rx: x is a reptile
Zx: x lives at the zoo
Lxy: x loves y
s: Amos
b: Bouncer
c: Cleo

1. Amos, Bouncer, and Cleo all live at the zoo.
2. Bouncer is a reptile, but not an alligator.
3. If Cleo loves Bouncer, then Bouncer is a monkey.
4. If both Bouncer and Cleo are alligators, then Amos loves them both.
5. Some reptile lives at the zoo.
6. Every alligator is a reptile.
7. Any animal that lives at the zoo is either a monkey or an alligator.
8. There are reptiles which are not alligators.
9. Cleo loves a reptile.
10. Bouncer loves all the monkeys that live at the zoo.
11. All the monkeys that Amos loves love him back.
12. If any animal is a reptile, then Amos is.

13. If any animal is an alligator, then it is a reptile.
14. Every monkey that Cleo loves is also loved by Amos.
15. There is a monkey that loves Bouncer, but sadly Bouncer does not reciprocate this love.

Part F These are syllogistic figures identified by Aristotle and his successors, along with their medieval names. Translate each argument into FOL.

Baralipton All *B*s are *C*s. All *A*s are *B*s. ∴ Some *C* is *A*.

Barbara All *B*s are *C*s. All *A*s are *B*s. ∴ All *A*s are *C*s.

Baroco All *C*s are *B*s. Some *A* is not *B*. ∴ Some *A* is not *C*.

Bocardo Some *B* is not *C*. All *A*s are *B*s. ∴ Some *A* is not *C*.

Celantes No *B*s are *C*s. All *A*s are *B*s. ∴ No *C*s are *A*s.

Celarent No *B*s are *C*s. All *A*s are *B*s. ∴ No *A*s are *C*s.

Cemestres No *C*s are *B*s. No *A*s are *B*s. ∴ No *A*s are *C*s.

Cesare No *C*s are *B*s. All *A*s are *B*s. ∴ No *A*s are *C*s.

Dabitis All *B*s are *C*s. Some *A* is *B*. ∴ Some *C* is *A*.

Darii All *B*s are *C*s. Some *A* is *B*. ∴ Some *A* is *C*.

Datisi All *B*s are *C*s. Some *B* is *A*. ∴ Some *C* is *A*.

Disamis Some *A* is *B*. All *A*s are *C*s. ∴ Some *B* is *C*.

Ferison No *B*s are *C*s. Some *B* is *A*. ∴ Some *A* is not *C*.

Ferio No *B*s are *C*s. Some *A* is *B*. ∴ Some *A* is not *C*.

Festino No *C*s are *B*s. Some *A* is *B*. ∴ Some *A* is not *C*.

Frisesomorum Some *B* is *C*. No *A*s are *B*s. ∴ Some *C* is not *A*.

Part G Using the symbolization key given, translate each English-language sentence into FOL.

UD: All animals
***Dx*:** *x* is a dog
***Sx*:** *x* likes samurai movies
***Lxy*:** *x* is larger than *y*
***b*:** Bertie
***e*:** Emerson
***f*:** Fergis

1. Bertie is a dog who likes samurai movies.

2. Bertie, Emerson, and Fergis are all dogs.
3. Emerson is larger than Bertie, and Fergis is larger than Emerson.
4. All dogs like samurai movies.
5. Only dogs like samurai movies.
6. There is a dog that is larger than Emerson.
7. If there is a dog larger than Fergis, then there is a dog larger than Emerson.
8. No animal that likes samurai movies is larger than Emerson.
9. No dog is larger than Fergis.
10. Any animal that dislikes samurai movies is larger than Bertie.
11. There is an animal that is between Bertie and Emerson in size.
12. There is no dog that is between Bertie and Emerson in size.
13. No dog is larger than itself.

Part H For each argument, write a symbolization key and translate the argument into FOL.

1. Nothing on my desk escapes my attention. There is a computer on my desk. As such, there is a computer that does not escape my attention.
2. All my dreams are black and white. Old TV shows are in black and white. Therefore, some of my dreams are old TV shows.
3. Neither Holmes nor Watson has been to Australia. A person could see a kangaroo only if they had been to Australia or to a zoo. Although Watson has not seen a kangaroo, Holmes has. Therefore, Holmes has been to a zoo.
4. No one expects the Spanish Inquisition. No one knows the troubles I've seen. Therefore, anyone who expects the Spanish Inquisition knows the troubles I've seen.
5. All babies are illogical. Nobody who is illogical can manage a crocodile. Berthold is a baby. Therefore, Berthold is unable to manage a crocodile.

Part I Using the symbolization key given, translate each English-language sentence into FOL.

UD: Candies
Cx**:** x has chocolate in it
Mx**:** x has marzipan in it
Sx**:** x has sugar in it
Txy**:** x has tried y
Bxy**:** x is better than y
b**:** Boris

1. Boris has never tried any candy.
2. Marzipan is always made with sugar.
3. Some candy is sugar-free.
4. No candy is better than itself.
5. Boris has never tried sugar-free chocolate.

6. Boris has tried marzipan and chocolate, but never together.

Part J Using the symbolization key given, translate each English-language sentence into FOL.

UD: People and dishes at a potluck
Rx: x has run out
Tx: x is on the table
Fx: x is food
Px: x is a person
Lxy: x likes y
e: Eli
f: Francesca
g: the guacamole

1. All the food is on the table.
2. If the guacamole has not run out, then it is on the table.
3. Everyone likes the guacamole.
4. If anyone likes the guacamole, then Eli does.
5. Francesca only likes the dishes that have run out.
6. Francesca likes no one, and no one likes Francesca.
7. Eli likes anyone who likes the guacamole.

Part K Using the symbolization key given, translate each English-language sentence into FOL.

UD: People
Dx: x dances ballet
Fx: x is female
Mx: x is male
Cxy: x is a child of y
Sxy: x is a sibling of y
e: Elmer
j: Jane
p: Patrick

1. All of Patrick's children are ballet dancers.
2. Jane is Patrick's daughter.
3. Patrick has a daughter.
4. Jane is an only child.
5. All of Patrick's daughters dance ballet.
6. Patrick has no sons.

Part L Identify which variables are bound and which are free.

1. $(\exists x\, Lxy \land \forall y\, Lyx)$

2. $(\forall x\, Ax \land Bx)$
3. $\big(\forall x\,(Ax \land Bx) \land \forall y\,(Cx \land Dy)\big)$
4. $\big[\forall x\,\exists y\big(Rxy \to (Jz \land Kx)\big) \lor Ryx\big]$
5. $\big(\forall x\,(My \leftrightarrow Lyx) \land \exists y\, Lzy\big)$

Chapter 8

Proofs involving universal quantifiers

In this chapter, we will discuss the natural deduction rules associated with the universal quantifier. We will look at the rules for the existential quantifiers in the next chapter. One rule will be easy; the other will be more difficult to apply. Before we get to these rules, we will clarify some of the terminology involved.

It will help to have an example to work with. So throughout this chapter we will assume the following symbolization key:

UD: The people at a certain party
***Dx*:** x is dancing
***Cx*:** x is chatting
***Lxy*:** x loves y
***a*:** Ashni
***b*:** Ben

8.1 Terminology

Instances

Suppose you start with a universal generalization (that is a statement that starts with '$\forall x$', or '$\forall y$', or the like). You then remove the initial quantifier and the variable, and replace every occurrence of that variable in the statement with a name or constant—the same name or constant each time. The result is an *instance* (or a substitution instance) of the generalization. For example:

A universal generalization . . .	**. . . and an instance**
$\forall x\, Dx$ (Everyone is dancing.)	Da (Ashni is dancing.)
$\forall y\, (Dy \land Cy)$ (Everyone is dancing and chatting.)	$(Db \land Cb)$ (Ben is dancing and chatting.)
$\forall z\, (Dz \to Cz)$ (Everyone who is is dancing is also chatting.)	$(Da \to Ca)$ (If Ashni is dancing she is also chatting.)

To speak loosely, a universal generalization makes a claim about everything in the universe of discourse (UD); an instance makes that claim about one specific thing. I hope you find it obvious that from a universal generalization you can validly infer any one of its instances: this in brief is the elimination rule for the universal quantifier. More on this below.

In what follows, we will let '$\forall \chi\, \mathcal{A}\chi$' stand for any arbitrary universal generalization, and '$\mathcal{A}c$' for one of its instances.

Discharging assumptions

Recall that when a subproof ends, the assumption of that subproof (i.e., the statement with which the subproof began) is said to be 'discharged'. Consider for example this proof:

```
1 │ (Pa → a = b)
2 │ (a = b → Rab)
  ├──────────────
3 │ │ Pa
  │ ├────
4 │ │ a = b          →E 3, 1
5 │ │ Rab            →E 2, 4
6 │ (Pa → Rab)       →I 3–5
```

In this case, the assumption Pa is introduced at line 3, and then discharged immediately after line 5. An assumption is said to be 'undischarged' if it has been introduced and not yet discharged. This includes the initial premises of the proof. So for example, at line 4 of the above proof, the undischarged assumptions are $(Pa \to a = b)$, $(a = b \to Rab)$ and Pa. At line 6, the undischarged assumptions are just $(Pa \to a = b)$ and $(a = b \to Rab)$.

8.2 Universal elimination

We first start with the easier rule for the universal quantifier: universal elimination. If you have $\forall x\, Px$, it is legitimate to infer that anything in the UD is a P. You can infer Pa, Pb, Pu, Pc_{21}, and so forth (assuming, of course, that a, b, u, and c_{21} name things in the UD). You can infer any substitution instance.

This is the general form of the universal elimination rule ($\forall$E):

$$\begin{array}{r|ll} j & \forall \mathscr{x}\, \mathscr{A}\mathscr{x} & \\ k & \mathscr{A}c & \forall\text{E } j \end{array}$$

When using the $\forall$E rule, you write the substituted sentence with the constant c replacing *all* occurrences of the variable $\mathscr{x}$ in $\mathscr{A}$. For example:

$$\begin{array}{r|ll} 1 & \forall x\,(Dx \rightarrow Lxa) & \\ \cline{2-2} 2 & (Db \rightarrow Lba) & \forall\text{E } 1 \\ 3 & (Da \rightarrow Laa) & \forall\text{E } 1 \end{array}$$

8.3 Universal introduction

A universal claim like $\forall x\, Px$ would be proven if every substitution instance of it had been proven. That is, if every sentence Pa, Pb, … were available in a proof, then you would certainly be entitled to claim $\forall x\, Px$. Alas, in most cases there is no hope of proving *every* substitution instance.

Consider instead a simple proof:

$$\begin{array}{r|l|ll} 1 & \multicolumn{2}{l}{\forall x\,(Cx \rightarrow Dx)} & \\ \cline{2-3} 2 & & (Ce \wedge Lea) & \\ \cline{3-3} 3 & & Ce & \wedge\text{E } 2 \\ 4 & & (Ce \rightarrow De) & \forall\text{E } 1 \\ 5 & & De & \rightarrow\text{E } 3, 4 \\ 6 & \multicolumn{2}{l}{\big((Ce \wedge Lea) \rightarrow De\big)} & \rightarrow\text{I } 2\text{–}5 \end{array}$$

In this proof, we have proven $\big((Ce \wedge Lea) \rightarrow De\big)$. However, using the very same method we could prove $\big((Cf \wedge Lfa) \rightarrow Df\big)$. Or we could have proven

$\big((Cg \wedge Lga) \to Dg\big)$. Indeed, we have a general method for proving *any* instance of the universal generalization $\forall x((Cx \wedge Lxa) \to Dx)$. But surely this is enough to justify the universal generalization itself! If all instances of the universal generalization can be proven, this means that all of them must be true (given the truth of our premise), which means that the universal generalization itself must be true (again, given the truth of our premise).

Thus, we can reason in the following way:

1	$\forall x\,(Cx \to Dx)$	
2	$\quad (Ce \wedge Lea)$	
3	$\quad Ce$	$\wedge$E 2
4	$\quad (Ce \to De)$	$\forall$E 1
5	$\quad De$	$\to$E 3, 4
6	$\big((Ce \wedge Lea) \to De\big)$	$\to$I 2–5
7	$\forall x\,\big((Cx \wedge Lxa) \to Dx\big)$	$\forall$I 6

Now, it is *crucial* to this proof that e was just some arbitrary constant: it stand for any object in the UD. To make sure we're dealing with an arbitrary constant, we pick a constant that is not already in use in the proof. That way, we guarantee that we are not making any special assumption about it.

To see this point, consider:

1	$\forall x\,Lxa$	
2	Laa	$\forall$E 1
3	$\forall x\,Lxx$	$\forall$I 2

Obviously there must be *something* wrong with this proof, because 'Everyone loves themself' cannot be validly inferred from 'Everyone loves Ashni'. But where is the error? The answer is that the final step in the proof is wrong. Because the letter a appears in the argument's premise, the $\forall I$ rule cannot be applied to it.

This is the schematic form of the universal introduction rule ($\forall$I):

j	$\mathcal{A}c^{*}$	
k	$\forall \chi\,\mathcal{A}\chi$	$\forall$I j

* The constant c must not occur in any undischarged assumption.

Note that we can do this for any constant that does not occur in an undischarged assumption and for any variable.

Note also that while the constant may not occur in any *undischarged* assumption, it may occur as the assumption of a subproof that we have already closed. For example, we can prove $\forall z\,(Cz \to Cz)$ without any premise:

1	Cf	
2	Cf	R 1
3	$(Cf \to Cf)$	$\to$I 1–2
4	$\forall z\,(Cz \to Cz)$	$\forall$I 3

Summary of derivation rules in FOL covered in this chapter

Universal Introduction ($\forall$I)

j	$\mathscr{A}c^{*}$	
k	$\forall\chi\,\mathscr{A}\chi$	$\forall$I j

* The constant c must not occur in any undischarged assumption.

Universal Elimination ($\forall$E)

j	$\forall\chi\,\mathscr{A}\chi$	
k	$\mathscr{A}c$	$\forall$E j

Practice exercises

Part A [*] Identify the mistake in each of the following "proofs".

1.

$$\begin{array}{r|ll}
1 & \forall x\,(Mx \to Fx) & \\
2 & \forall x\,(Fx \to Px) & \\
\hline
3 & (Mx \to Fx) & \forall\text{E } 1 \\
4 & (Fx \to Px) & \forall\text{E } 1 \\
5 & \quad \vert\ \underline{Mx} & \\
6 & \quad \vert\ Fx & \to\text{E } 3, 5 \\
7 & \quad \vert\ Px & \to\text{E } 4, 6 \\
8 & (Mx \to Px) & \to\text{I } 5\text{–}7 \\
9 & \forall x\,(Mx \to Px) & \forall\text{I } 8
\end{array}$$

2.

$$\begin{array}{r|ll}
1 & (\forall x\, Px \to Ga) & \\
2 & Pb & \\
3 & \forall x\, Px & \\
\hline
4 & (Pb \to Ga) & \forall\text{E } 1 \\
5 & Ga & \to\text{E } 2, 4
\end{array}$$

---

[*] Many thanks to Kesavan Thanagopal for offering these exercises, originally created for his PHIL110 *Introduction to Logic and Reasoning* tutorials.

3.

1	$\forall x\,(Px \wedge Qx)$	
2	Ra	
3	$(Pa \wedge Qa)$	$\forall$E 1
4	Qa	$\wedge$E 3
5	$\forall x\, Qx$	$\forall$I 4

Part B Provide natural deduction proofs for the following inferences:

1. $\forall x\,(Cx \to Dx)$, Ca
   $\therefore\ Da$

2. $\forall x\big((Cx \wedge Lxa) \to Dx\big)$, Cb, Lba
   $\therefore\ Db$

3. $\forall x\, Cx$, $\forall x\, Dx$
   $\therefore\ (Ca \wedge Da)$

Part C Provide natural deduction proofs for the following inferences:

1. $\forall x\,(Cx \wedge Dx)$
   $\therefore\ \forall\, xCx$

2. $\forall x\,(Cx \to Dx)$, $\forall x\, Cx$
   $\therefore\ \forall\, xDx$

3. $\forall x\,(Cx \to Dx)$
   $\therefore\ \forall x\,(\neg Dx \to \neg Cx)$

Part D Some of the following inferences are valid, some are not. For each, either provide a formal proof in our formal system or construct an FO counterexample. (*challenging*)

1. $\forall x\,(Px \vee Qx)$
   $\therefore\ (\forall x\, Px \vee \forall x\, Qx)$

2. $\forall x\,(Px \wedge Qx)$
   $\therefore\ (\forall x\, Px \wedge \forall x\, Qx)$

3. $(\forall x\, Px \lor \forall x\, Qx)$
   $\therefore\ \forall x\, (Px \lor Qx)$

4. $(\forall x\, Px \land \forall x\, Qx)$
   $\therefore\ \forall x\, (Px \land Qx)$

Chapter 9

Proofs involving existential quantifiers

In this chapter, we will discuss the natural deduction rules associated with the existential quantifier. To keep things simple, we shall use the same symbolization key from the previous chapter:

UD: The people at a certain party
Dx: x is dancing
Cx: x is chatting
Lxy: x loves y
a: Ashni
b: Ben

As with universal generalizations, existential generalizations—i.e., statements that start with ‘$\exists x$’, or ’$\exists z$’, or the like—have instances. To find an instance of an existential generalization, you remove the initial quantifier and variable, and replace every occurrence of that variable in the statement with a name or constant—the same name or constant each time. For example:

An existential generalization . . .	**. . . and an instance**
$\exists x\, Dx$ (Someone is dancing.)	Da (Ashni is dancing.)
$\exists y\, (Dy \wedge Cy)$ (Someone is dancing and chatting.)	$(Db \wedge Cb)$ (Ben is dancing and chatting.)
$\exists z\, (Dz \vee Cz)$ (Someone is either dancing or chatting.)	$(Da \vee Ca)$ (Ashni is either dancing or chatting.)

In what follows, '$\exists x\,\mathcal{A}\chi$' is an existential generalization and '$\mathcal{A}c$' is an instance.

One rule for the existential quantifier is easy to apply; the other is more difficult. We begin with the easier rule.

9.1 Existential introduction

It is legitimate to infer $\exists x\, Px$ if you know that *something* is a P. It might be any particular thing at all. For example, if you have Pa available in the proof, then $\exists x\, Px$ follows.

This is the existential introduction rule ($\exists$I):

j	$\mathcal{A}c$	
k	$\exists\chi\,\mathcal{A}\chi$	$\exists$I j

It is important to notice that the variable x does not need to replace all occurrences of the constant c. You can decide which occurrences to replace and which to leave in place. For example:

1	$Da \to Lad$	
2	$\exists x\,(Da \to Lax)$	$\exists$I 1
3	$\exists x\,(Dx \to Lxd)$	$\exists$I 1
4	$\exists x\,(Dx \to Lad)$	$\exists$I 1
5	$\exists y\,\exists x\,(Dx \to Lyd)$	$\exists$I 4
6	$\exists z\,\exists y\,\exists x\,(Dx \to Lyz)$	$\exists$I 5

Sentences on lines 5 and 6 are examples of what is called 'multiple quantification': sentences that contain a series of two or more quantifiers in a row. We will look at multiple quantification more closely in the next chapter. What is important to note here is that the inferences made at lines 5 and 6 are valid.

9.2 Existential elimination

Recall that a sentence with an existential quantifier tells us that there is some member of the UD that satisfy a formula. For example, $\exists x\, Sx$ tells us (roughly) that there is at least one S. It does not tell us *which* member of the UD satisfies S, however. We cannot immediately conclude Sa, Sf_{23}, or any other substitution instance of the sentence. What can we do?

Suppose that we knew both $\exists x\, Sx$ and $\forall x\, (Sx \rightarrow Tx)$. We could reason in this way:

> Since $\exists x\, Sx$, there is something that is an S. We do not know which constants refer to this thing, if any do, so call this thing 'Ishmael'. From $\forall x\, (Sx \rightarrow Tx)$, it follows that if Ishmael is an S, then it is a T. Therefore, Ishmael is a T. Because Ishmael is a T, we know that $\exists x\, Tx$.

In this paragraph, we introduced a name for the thing that is an S. We gave it an arbitrary name ('Ishmael') so that we could reason about it and derive some consequences from there being an S. Since 'Ishmael' is just a bogus name introduced for the purpose of the proof and not a genuine constant, we could not mention it in the conclusion. Yet we could derive a sentence that does not mention Ishmael; namely, $\exists x\, Tx$. This sentence does follow from the two premises.

We want the existential elimination rule to work in a similar way. Yet, since English language words like 'Ishmael' are not symbols of FOL, we cannot use them in formal proofs. Instead, we will use constants of FOL which do not otherwise appear in the proof.

A constant that is used to stand in for whatever it is that satisfies an existential claim is called a 'proxy'. Reasoning with the proxy must all occur inside a subproof, and the proxy cannot be a constant that is doing work elsewhere in the proof.

This is the schematic form of the existential elimination rule ($\exists$E):

$$\begin{array}{r|l l}
j & \exists \chi\, \mathcal{A}\chi & \\
k & \quad \mathcal{A}c^{*} & \\
l & \quad \mathcal{B} & \\
m & \mathcal{B} & \exists\text{E } j, k\text{–}l
\end{array}$$

* The constant c must not appear in $\exists \chi\, \mathcal{A}\chi$, in $\mathcal{B}$, or in any undischarged assumption.

Since the proxy constant is just a placeholder that we use inside the subproof, it cannot be something about which we know anything particular. So, it cannot appear in the original sentence $\exists x\, Ax$ or in an undischarged assumption. Moreover, we do not learn anything about the proxy constant by using the $\exists$E rule. So it cannot appear in B, the sentence you prove using $\exists$E.

With this rule, we can give a formal proof that $\exists x\, Sx$ and $\forall x\, (Sx \rightarrow Tx)$ together entail $\exists x\, Tx$.

1	$\exists x\, Sx$	
2	$\forall x\, (Sx \rightarrow Tx)$	
3	$\quad Si$	
4	$\quad (Si \rightarrow Ti)$	$\forall$E 2
5	$\quad Ti$	$\rightarrow$E 3, 4
6	$\quad \exists x\, Tx$	$\exists$I 5
7	$\exists x\, Tx$	$\exists$E 1, 3–6

Notice that this has effectively the same structure as the English-language argument with which we began, except that the subproof uses the proxy constant '*i*' rather than the bogus name 'Ishmael'.

9.3 Quantifier equivalences

Back in chapter 7, we said that $\forall \chi \neg \mathcal{A}$ and $\neg \exists \chi\, \mathcal{A}$ are logically equivalent. We also said that $\exists \chi \neg \mathcal{A}$ and $\neg \forall \chi\, \mathcal{A}$ were logically equivalent. If two sentences are logically equivalent, then we can derive one from the other, and vice versa. We are now in position to prove these first-order equivalences. If we use the wff Px as our $\mathcal{A}$, then showing these equivalences would amount to proving the following four inferences:

1. $\exists x \neg Px \quad \therefore \quad \neg \forall x\, Px$
2. $\neg \forall x\, Px \quad \therefore \quad \exists x \neg Px$
3. $\forall x \neg Px \quad \therefore \quad \neg \exists x\, Px$
4. $\neg \exists x\, Px \quad \therefore \quad \forall x \neg Px$

We are going to prove two of these inferences below (i.e., 1 and 3). The other two will be left as exercises.

Let's begin with 1. We have an existential claim as premise. This is a good indication that we will need to use the $\exists$ E rule. Let's see if we can use it to get our conclusion. According to the rule, we could derive $\neg \forall x\, Px$ from $\exists x \neg Px$ from the assumption that some proxy (say e) is the object that's not a P:

1	$\exists x \neg Px$	
2	$\quad \neg Pe$	
	$\quad \ldots$	
?	$\quad \neg \forall x\, Px$	
	$\neg \forall x\, Px$	∃E 1, 2–?

Now let's focus on $\neg \forall x\, Px$ that's within the subproof. It's a negation, so we can use ¬I to justify it:

1	$\exists x \neg Px$	
2	$\quad \neg Pe$	
3	$\quad\quad \forall x\, Px$	
??	$\quad\quad \ldots$	
?	$\quad \neg \forall x\, Px$	¬I 3–??
	$\neg \forall x\, Px$	∃E 1, 2–?

Of course, in order to use the ¬I rule, we need to find some sentence and its negation. That's easy, since it follows from line 3 that *Pe*, which contradicts the sentence at line 2. The finished proof looks like this:

1	$\exists x \neg Px$	
2	$\quad \neg Pe$	
3	$\quad\quad \forall x\, Px$	
4	$\quad\quad Pe$	∀E 3
5	$\quad\quad \neg Pe$	R 2
6	$\quad \neg \forall x\, Px$	¬I 3–5
7	$\neg \forall x\, Px$	∃E 1, 2–6

Now let's prove that $\neg \exists x\, Px$ follows from $\forall x \neg Px$. Since we're trying to establish a negation, that is a good indication that our proof will take the form of a *reductio*:

1	$\forall x \neg Px$	
2	$\exists x\, Px$	
?	$\dots$	
	$\neg\exists x\, Px$	$\neg$I 2–?

In order to properly use the ¬I rule, we need to end the subproof with a sentence and its negation. So, if we can get $\neg\forall x \neg Px$ in our proof, that would be the negation of line 1:

1	$\forall x \neg Px$	
2	$\exists x\, Px$	
	$\dots$	
	$\neg\forall x \neg Px$	
?	$\forall x \neg Px$	R 1
	$\neg\exists x\, Px$	$\neg$I 2–?

Our next task is to establish $\neg\forall x \neg Px$. How can we do that? We can try using ∃E on line 2:

1	$\forall x \neg Px$	
2	$\exists x\, Px$	
3	Pa	
	$\dots$	
??	$\neg\forall x \neg Px$	
	$\neg\forall x \neg Px$	$\exists$E 2, 3–??
?	$\forall x \neg Px$	R 1
	$\neg\exists x\, Px$	$\neg$I 2–?

Now we need to establish $\neg\forall x \neg Px$ again. How do we do that? Again, we can try using *reductio*:

1	│ $\forall x\,\neg Px$	
2	│ │ $\exists x\,Px$	
3	│ │ │ Pa	
4	│ │ │ │ $\forall x\,\neg Px$	
???	│ │ │ │ $\dots$	
??	│ │ │ $\neg\forall x\,\neg Px$	¬I 4–???
	│ │ $\neg\forall x\,\neg Px$	∃E 2, 3–??
?	│ │ $\forall x\,\neg Px$	R 1
	│ $\neg\exists x\,Px$	¬I 2–?

We simply now need to get some sentence and its negation. We can do this rather easily, and complete our proof:

1	│ $\forall x\,\neg Px$	
2	│ │ $\exists x\,Px$	
3	│ │ │ Pa	
4	│ │ │ │ $\forall x\,\neg Px$	
5	│ │ │ │ $\neg Pa$	∀E 4
6	│ │ │ │ Pa	R 3
7	│ │ │ $\neg\forall x\,\neg Px$	¬I 4–6
8	│ │ $\neg\forall x\,\neg Px$	∃E 2, 3–7
9	│ │ $\forall x\,\neg Px$	R 1
10	│ $\neg\exists x\,Px$	¬I 2–9

9.4 Soundness and completeness for FOL

We have now completed constructing our formal system of natural deduction in FOL. Our system contains the introduction and elimination rules for the truth-functional connectives, the quantifiers, and identity, as well as the reiteration rule: 17 rules in all. At this point, we might ask ourselves two important questions about our proof system. First, we might wonder whether our proof system leaves anything out: are there first-order consequences that cannot be proven using our proof system? Second, we might wonder whether our proof system makes any mistake. Surely, *we* can make mistakes using our proof system. But are there things that can be proven in our system that are not actually first-order consequences?

Our proof system warrants negative answers to both questions, which lead us to two desirable properties of a system of natural deduction. The first property is *completeness* (note that this is different from the notion of truth-functional completeness discussed in chapter 4). Our first-order system of natural deduction is able to prove *every* first-order consequence. It doesn't leave anything out. If we let '$P_1, \ldots, P_n \vdash S$' mean that a sentence S is provable in FOL from premises P_1–P_n, we can express the completeness theorem like this:

Completeness Theorem: If S is a first-order consequence of $P_1, \ldots, P_n$, then $P_1, \ldots, P_n \vdash S$.

The second desirable property is *soundness* (note that this is different from the notion of a sound argument). Our first-order proof system *only* proves first-order consequences. That is, it doesn't make mistakes. Of course, *we* might make mistakes in using the rules, but if we do, the problem is with us, not with our system. Again, if we let '$P_1, \ldots, P_n \vdash S$' mean that a sentence S is provable in FOL from premises P_1–P_n, we can express the soundness theorem like this:

Soundness Theorem: If $P_1, \ldots, P_n \vdash S$, then S is a first-order consequence of $P_1, \ldots, P_n$.

Taken together, the soundness and completeness theorems tell us that our proof system proves all and only first-order consequences:

$P_1, \ldots, P_n \vdash S$ if and only if S is a first-order consequence of $P_1, \ldots, P_n$.

We do not have all of the technical resources to prove the completeness theorem for FOL. But we can go some way to provide a sketch of a proof for the soundness theorem.

To see how this might go—and to make our job a little easier—consider a *subset* of our formal system, one that only contains the introduction and elimination rules for the truth-functional connectives. Call that system "t". There is a corresponding completeness to t:

Soundness Theorem$_t$: If $P_1, \ldots, P_n \vdash_t S$, then S is tautological consequence of $P_1, \ldots, P_n$.

Here, the subscript 't' is to make it clear that we are talking about a subset of our first-order proof system that includes only the introduction and elimination rules for the truth-functional connectives. So, '$P_1, \ldots, P_n \vdash_t S$' means that S is provable from sentences $P_1, \ldots, P_n$ just using the rules for the truth-functional connectives. And recall from chapter 3 that a sentence S is a tautological consequence of sentences $P_1, \ldots, P_n$ just in case there is no row in a joint truth table that assigns a T to all of $P_1, \ldots, P_n$ and an F to S.

The proof of the soundness of t takes the form of a *reductio ad absurdum*. First, we assume for the sake of contradiction that t is unsound. So, we assume that there is a proof p in t that makes a mistake and that allows us to derive a sentence S from given premises, but where S is not a tautological consequence of the premises. p might contain more than one mistake, but all we need to consider is the first supposed invalid step in p. The next step in establishing the soundness theorem for t involves showing that none of the 10 rules of t can be responsible for this first invalid step. This can take a bit of time, since we need to treat each rule separately, but the idea is that once we've done this, we will have shown that t doesn't make any mistake, thus establishing the soundness theorem for t.

Let us end this chapter by looking at two examples: we will show that neither $\wedge$I nor $\neg$E can be responsible for the first invalid step in p.

First, suppose that the first invalid step derives the sentence $(\mathcal{P} \wedge \mathcal{Q})$ from an application of $\wedge$I to $\mathcal{P}$ and $\mathcal{Q}$. Let $\mathcal{A}_1, \ldots, \mathcal{A}_k$, $\mathcal{P}$, and $\mathcal{Q}$ be the assumptions in force at $(\mathcal{P} \wedge \mathcal{Q})$. Here, we are assuming that $(\mathcal{P} \wedge \mathcal{Q})$ is not a tautological consequence of $\mathcal{A}_1, \ldots, \mathcal{A}_k$. Since the steps deriving $\mathcal{P}$ and $\mathcal{Q}$ are before the first invalid step, $\mathcal{P}$ and $\mathcal{Q}$ must be tautological consequences of $\mathcal{A}_1, \ldots, \mathcal{A}_k$. Now imagine constructing a joint truth table for $A_1, \ldots, \mathcal{A}_k$, $\mathcal{P}$, $\mathcal{Q}$, and $(\mathcal{P} \wedge \mathcal{Q})$. Since we're assuming that $(\mathcal{P} \wedge \mathcal{Q})$ is not a tautological consequence of $\mathcal{A}_1, \ldots, \mathcal{A}_k$, this means that there must be a row in the joint truth table that assigns a T to all of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ and an F to $(\mathcal{P} \wedge \mathcal{Q})$. Call this row '$h$'. Since $(\mathcal{P} \wedge \mathcal{Q})$ is false in h, so is $\mathcal{P}$ and so is $\mathcal{Q}$. But now we've reached a contradiction: $\mathcal{P}$ and $\mathcal{Q}$ must both be true in h, since we're assuming that they are both tautological consequences of $\mathcal{A}_1, \ldots, \mathcal{A}_k$. So, contrary to our initial assumption, $\wedge$I cannot be responsible for the first invalid step.

Next, suppose the first invalid step in p derives the sentence $\mathcal{P}$ from an application of $\neg$E to an earlier subproof with assumption $\neg\mathcal{P}$ and that ends with $\mathcal{Q}$ and $\neg\mathcal{Q}$ (a contradiction). Again, let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be the assumptions in force at step $\mathcal{P}$. Note that the assumptions in force at step $\neg\mathcal{Q}$ are $\mathcal{A}_1, \ldots, \mathcal{A}_k$, $\neg\mathcal{P}$, and $\mathcal{Q}$. The assumptions in force at step $\mathcal{Q}$ are $\mathcal{A}_1, \ldots, \mathcal{A}_k$ and $\neg\mathcal{P}$. Since the steps deriving $\mathcal{Q}$ and $\neg\mathcal{Q}$ are earlier than our first invalid step, $\mathcal{Q}$ and $\neg\mathcal{Q}$ must be tautological consequences of $\mathcal{A}_1, \ldots, \mathcal{A}_k$, and $\neg\mathcal{P}$. However, the only way for $\mathcal{Q}$ and $\neg\mathcal{Q}$ to both be tautological consequences of $\mathcal{A}_1, \ldots, \mathcal{A}_k$, and $\neg\mathcal{P}$ is for the set of premises to be tautologically inconsistent: either $\mathcal{A}_1, \ldots, \mathcal{A}_k$ or $\neg\mathcal{P}$ must be false. Now, imagine constructing a joint truth table for $\mathcal{A}_1, \ldots, \mathcal{A}_k$, $\mathcal{P}$, $\mathcal{Q}$, $\neg\mathcal{Q}$, and $\neg\mathcal{P}$. Since we're assuming that $\mathcal{P}$ is not a tautological consequence of $\mathcal{A}_1$, $\ldots, \mathcal{A}_k$, this means that there must be a row in the joint truth table that assigns a T to all of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ and an F to $\mathcal{P}$. Call this row 'h'. In h, $\mathcal{A}_1, \ldots, \mathcal{A}_k$ and $\neg\mathcal{P}$ are true. But that contradicts our earlier result that either $\mathcal{A}_1, \ldots, \mathcal{A}_k$ or $\neg\mathcal{P}$ must be false. So, contrary to our initial assumption, $\neg$E cannot be responsible for the first invalid step.

The full proof of the soundness of t requires showing that the other 8 rules cannot also be responsible for the first invalid step. Similarly, the full proof for the soundness for FOL requires showing that none of the 17 rules is responsible for the first invalid step. (Although here we would need to move away from the notion of *tautological consequence* and frame the proof in terms of 'first-order consequence', and part of what makes our task difficult is that we don't have a proper definition of a first-order consequence.)

9.5 Proving invalidity

There are times when working through a difficult proof where we get stuck and are unsure how best to proceed. But given that our proof system is *complete*, we know that if what we're trying to derive is in fact a logical consequence of the premises, then there exists a proof in our proof system that will show this. We can rest assured that with more time and practice, we'll be able to work our way to a completed proof.

Other times, however, the reason why we might be struggling to complete a 'proof' is that the inference is in fact logically invalid. Given that our proof system is *sound*, this means that there can be no such proof in our proof system. Trying to work out such a 'proof' would be in vain.

How can we prove that an inference is logically invalid? What we do is construct what is called a 'first-order (FO) counterexample'. An FO counterexample consists in providing a symbolization key for the predicates and constants used in the inference—except for the identity symbol, which always expresses identity—and describing a circumstance in which the premises are true and the conclusion is false.

For example, consider the following inference:

$$\forall x\,(Px \rightarrow Qx), \forall x\,(Qx \rightarrow Rx), \neg Pa \qquad \therefore \qquad \neg Ra$$

To show that this inference is invalid, consider first the following symbolization key:

UD: Everything
***Px*:** x is dog
***Qx*:** x is a canine
***Rx*:** x is a mammal
***a*:** Smoky

Now, while it is true that all dogs are canines and that all canines are mammals, suppose Smoky was my pet cat, and thus not a dog. In this case, all of the premise would be true, but the conclusion would be false; after all, cats are mammals too.

This constitutes an counterexample to the inference above, showing that it is invalid. More generally, if an inference is invalid, it will always be possible to conceive of an FO counterexample of this kind.

Summary of derivation rules in FOL covered in this chapter

Existential Introduction ($\exists$I)

j	$\mathcal{A}c$	
k	$\exists\chi\,\mathcal{A}\chi$	$\exists$I j

Existential Elimination ($\exists$E)

j	$\exists\chi\,\mathcal{A}\chi$		
k		$\mathcal{A}c^{*}$	
l		$\mathcal{B}$	
m	$\mathcal{B}$		$\exists$E j, k–l

* The constant c must not appear in $\exists\chi\,\mathcal{A}\chi$, in $\mathcal{B}$, or in any undischarged assumption.

Practice exercises

Part A [*] Identify the mistake in each of the following "proofs".

1.

1	$\exists x\, Px$	
2	$\exists y\, Py$	R 1

2.

1	$\forall x\,(Fx \to Gx)$	
2	$\exists x\, Fx$	
3	│ Fk	for existential instantiation
4	│ $(Fk \to Gk)$	∀E 1
5	│ Gk	→E 3, 4
6	Gk	∃E 2, 3–5
7	$\exists y\, Gy$	∃I 6

3.

1	$\exists x\,(Fa \to Sx)$	
2	│ Fa	for conditional proof
3	│ │ $(Fa \to Sa)$	for existential instantiation
4	│ │ Sa	→E 2, 3
5	│ │ $\exists y\, Sy$	∃I 4
6	│ $\exists y\, Sy$	∃E 1, 3–5
7	$(Fa \to \exists y\, Sy)$	→I 2–6

---

[*] Many thanks to Kesavan Thanagopal for offering these exercises, originally created for his PHIL110 *Introduction to Logic and Reasoning* tutorials.

4.

1	$\forall x\,(Rx \lor Cx)$	
2	$\exists x\,\neg Rx$	
3	$\neg Rk$	for existential instantiation
4	$(Rk \lor Ck)$	$\forall$E 1
5	Ck	$\lor$E 3, 4
6	$\exists x\,Cx$	$\exists$I 5
7	$\exists x\,Cx$	$\exists$E 3–6

Part B The following inferences are valid. In each case, provide a natural deduction proof:

1. $Da, \quad Ca$
   $\therefore\ \exists x\,(Cx \land Dx)$
2. Da
   $\therefore\ \exists x\,(Dx \lor Cx)$
3. $\exists x\,(Cx \land Dx)$
   $\therefore\ \exists x\,Cx$
4. $\exists x\,\neg Cx, \quad \forall x\,(Dx \lor Cx)$
   $\therefore\ \exists x\,Dx$

Part C Exactly one of these inferences is valid. Give a natural deduction proof for the valid inference, and explain why the other inference is not valid by offering an FO counterexample:

1. $\neg Da, \quad (\exists x\,Cx \to Da)$
   $\therefore\ \forall x\,\neg Cx$
2. $\forall x\,(Dx \to \neg Cx), \quad \neg Ca$
   $\therefore\ \neg Da$

Part D Provide proofs for the following two inferences:

1. $\neg\forall x\, Px \quad \therefore \quad \exists x\, \neg Px$
2. $\neg\exists x\, Px \quad \therefore \quad \forall x\, \neg Px$

Part E Some of the following inferences are valid, some are not. For each, either provide a formal proof in our formal system or construct an FO counterexample. (*challenging*)

1. $\exists x\, (Px \vee Qx)$
   $\therefore\ (\exists x\, Px \vee \exists x\, Qx)$
2. $(\exists x\, Px \vee \exists x\, Qx)$
   $\therefore\ \exists x\, (Px \vee Qx)$
3. $\exists x\, (Px \wedge Qx)$
   $\therefore\ (\exists x\, Px \wedge \exists x\, Qx)$
4. $(\exists x\, Px \wedge \exists x\, Qx)$
   $\therefore\ \exists x\, (Px \wedge Qx)$

Part F Show that the rule →I is sound in *t*.

Chapter 10

Multiple quantifiers

So far, we've been working with quantified sentences that contain only one quantifier. In this chapter, we look at how to translate quantified sentences that require more than one quantifier, and we look at proofs that contain quantified sentences with multiple quantifiers.

10.1 The four Aristotelian forms

Before we get into the business of translating sentences, it will be helpful to look at the four main sentence forms treated in Aristotle's logic.

1. All $\mathcal{P}$'s are $\mathcal{Q}$'s.
2. Some $\mathcal{P}$'s are $\mathcal{Q}$'s.
3. No $\mathcal{P}$'s are $\mathcal{Q}$'s.
4. Some $\mathcal{P}$'s are not $\mathcal{Q}$'s.

We saw in previous chapters ways in which we can translate these sentence forms in FOL. In the next two chapters, it will be useful to remind ourselves of their FOL translations:

1^*. $\forall x\,(\mathcal{P}x \to \mathcal{Q}x)$
2^*. $\exists x\,(\mathcal{P}x \land \mathcal{Q}x)$
3^*. $\forall x\,(\mathcal{P}x \to \neg\mathcal{Q}x)$ (or alternatively $\neg\exists x\,(\mathcal{P}x \land \mathcal{Q}x)$)
4^*. $\exists x\,(\mathcal{P}x \land \neg\mathcal{Q}x)$

This will be useful since the majority of quantified English sentences that we will look at in the next two chapters can easily be seen as belonging to one of these four Aristotelian forms. Once we've identified the form, translating sentences into FOL will simply be a matter of fleshing out the details.

10.2 Multiple uses of a single quantifier

We'll first focus on quantified sentences with multiple uses of a single quantifier—i.e., multiple uses of $\exists$ and multiple uses of $\forall$. We'll then consider sentences that use mixed quantifiers.

Consider the following symbolization key and the sentences that follow it:

UD: Chess pieces on a chess board
Hx: x is a bishop
Kx: x is a knight
Lx: x is black
Px: x is a pawn
Rx: x is a rook
Wx: x is white
Bxy: x is in back of y
Fxy: x is in front of y
Sxy: x is in the same row as y

5. All pawns are in front of every rook.
6. A bishop is in back of a knight.
7. No black pawn is in front of any knight.
8. Some black piece is not in front of some white piece.
9. Every bishop is in a different row from every other bishop.

Sentence 5 clearly takes the form 'All $\mathcal{P}$'s are $\mathcal{Q}$'s'. So, our translation will be of the form $\forall x\,(\ldots \rightarrow \ldots)$. We simply need to figure out our $\mathcal{P}$'s and $\mathcal{Q}$'s. In this case, $\mathcal{P}$ is simply $Px : \forall x\,(Px \rightarrow \ldots)$. What is our $\mathcal{Q}$? It is being in front of every rook. So, we need a way to express that: $\forall y\,(Ry \rightarrow Fxy)$. Put those two bits together, and you've got a correct translation of 5:

5*. $\forall x \big(Px \rightarrow \forall y\,(Ry \rightarrow Fxy)\big)$

Sentence 6 takes the form 'Some $\mathcal{P}$'s are $\mathcal{Q}$'s'. So, our translation will take the form $\exists x\,(\ldots \wedge \ldots)$. Again, we simply need to figure out our $\mathcal{P}$'s and $\mathcal{Q}$'s; in this case, our $\mathcal{P}$ is simply Hx: $\exists x\,(Hx \wedge \ldots)$. What is our $\mathcal{Q}$? It is being in back of a knight. We can express this using: $\exists y(Ky \wedge Bxy)$. Putting those things together we get:

6*. $\exists x \big(Hx \wedge \exists y\,(Ky \wedge Bxy)\big)$

Sentence 7 is a bit trickier, but the same kind of procedure should yield a correct translation. First, we notice that sentence 7 is of the form 'No $\mathcal{P}$'s are

$\mathcal{Q}$'s.' Second, we notice that our $\mathcal{P}$ is something complex: being a black pawn. So, sentence 6 can be partially translated as $\forall x\big((Lx \land Px) \to \ldots\big)$. Our $\mathcal{Q}$ is not being in front of any knight: $\forall y\,(Ky \to \neg Fxy)$. We can now finish the translation:

7*. $\forall x\big((Lx \land Px) \to \forall y\,(Ky \to \neg Fxy)\big)$

Alternatively, we could have translated 7 as $\neg\exists x\big((Lx \land Px) \land \exists y\,(Ky \land Fxy)\big)$. A correct translation for 8 is:

8*. $\exists x\big(Lx \land \neg\exists y\,(Wy \land Fxy)\big)$

It might be tempting to translate sentence 9 as $\forall x\big(Hx \to \forall y\,(Hy \to \neg Sxy)\big)$. But this would be a mistake. $\forall x\big(Hx \to \forall y\,(Hy \to \neg Sxy)\big)$ states that every bishop is in a different row from every bishop. But this sentence would be false even if there are only two bishops in two different rows, since each bishop would obviously be in the same row as itself. In order to express the claim that each bishop is in a different row from every *other* bishop, we will need to make it clear that we are talking about different bishops. In order to do that, we will need the inequality symbol. We can translate sentence 9 with this:

9*. $\forall x\big[Hx \to \forall y\big((Hy \land y \neq x) \to \neg Syx\big)\big]$

10.3 Mixed quantifiers

We now move on to consider sentences that mix universal and existential quantifiers together. Consider the following symbolization key and the sentences that follow it:

UD: People and dogs
***Dx*:** x is a dog
***Fxy*:** x is a friend of y
***Oxy*:** x owns y
***f*:** Fifi
***g*:** Gerald

10. Fifi is a dog.
11. Gerard is a dog owner.
12. Someone is a dog owner.
13. All of Gerald's friends are dog owners.

14. Every dog owner is the friend of a dog owner.

Sentence 10 is easy: Df. Sentence 11 can be paraphrased as 'There is a dog that Gerald owns'. This has the form of 'Some P's are Q's'. This can be translated as $\exists x\,(Dx \wedge Ogx)$.

Like sentence 11, sentence 12 takes the form 'Some P's are Q's'. It can be paraphrased as 'Some dog is owned by someone'. So, we can translate it as $\exists x\,(Dx \wedge \exists y\, Oyx)$.

Sentence 13 can be paraphrased as 'Every friend of Gerald is a dog owner'. It clearly is of the form 'All P's are Q's'. Translating the first part of this sentence, we get $\forall x\,(Fxg \rightarrow$ 'x is a dog owner'$)$. Again, it is important to recognize that 'x is a dog owner' is structurally just like sentence 11. Since we already used the variable 'x', we will need a different variable for the existential quantifier. Any other variable will do. Using 'z', sentence 13 can be translated as $\forall x\,\big(Fxg \rightarrow \exists z\,(Dz \wedge Oxz)\big)$.

Sentence 14 can be paraphrased as 'For any x that is a dog owner, there is a dog owner who is x's friend.' Partially translated, this becomes:

$$\forall x\,\big(x \text{ is a dog owner } \rightarrow \exists y\,(y \text{ is a dog owner } \wedge Fxy)\big)$$

Completing the translation, sentence 14 becomes:

$$\forall x\,\big(\exists z\,(Dz \wedge Oxz) \rightarrow \exists y\,(\exists z\,(Dz \wedge Oyz) \wedge Fxy)\big)$$

When symbolizing sentences with multiple quantifiers, it is best to proceed by small steps, recognizing their Aristotelian forms. Paraphrase the English sentence so that the logical structure is readily symbolized in FOL. Then translate piecemeal, replacing the dauting task of translating a long sentence with the simple task of translating shorter formulas.

10.4 Order of quantifiers and variables

When dealing with multiple instances of a single quantifier, the order of the quantifiers didn't matter. If we let Lxy mean 'x likes y', obviously $\forall x\,\forall y\, Lxy$ and $\forall y\,\forall x\, Lxy$ are logically equivalent: they both state that everybody likes everybody. This same is true about the order of variables. $\forall x\,\forall y\, Lxy$ is logically equivalent to $\forall x\,\forall y\, Lyx$. Again, both mean that everybody likes everybody.

But this is not the case when we're dealing with mixed quantifiers. Both the order of variables and the order of quantifiers matter a great deal. To see this, consider the following four sentences:

15. $\forall x\, \exists y\, Lxy$
16. $\exists y\, \forall x\, Lxy$
17. $\forall x\, \exists y\, Lyx$
18. $\exists y\, \forall x\, Lyx$

Assuming we are working with people as our UD, sentence 15 states that everyone likes someone. Sentence 16, on the other hand, states that there is some popular person that everyone likes. Statement 17 states that everyone is liked by someone, whereas sentence 18 states that someone likes everyone.

So, when we're dealing with mixed quantifiers, we must pay close attention to both the order of the variables, as well as the order of the quantifiers.

10.5 Proofs using multiple quantifiers

Doing proofs with multiple quantifiers is not really any different from doing proofs with sentences that contain only one quantifier. They might appear to be trickier, but if we apply our rules correctly, we shouldn't have any issue. Let's take a look at a couple of examples before we end this chapter.

Our FOL translation of sentence 5 above was $\forall x\big(Px \to \forall y\,(Ry \to Fxy)\big)$. There is another way to translate this sentence such that all quantifiers appear at the beginning of the sentence: $\forall x\, \forall y\big((Px \land Ry) \to Fxy\big)$. It is said of the latter sentence that it is in *prenex* form, but it is logically equivalent to the former translation. Since they are logically equivalent, we can derive one from the other, and vice versa. Here, let's show that $\forall x\, \forall y\big((Px \land Ry) \to Fxy\big)$ follows from $\forall x\big(Px \to \forall y\,(Ry \to Fxy)\big)$.

$$\begin{array}{r|l}
1 & \forall x\big(Px \to \forall y\,(Ry \to Fxy)\big) \\ \cline{2-2}
 & \ldots \\
 & \forall x\, \forall y\big((Px \land Ry) \to Fxy\big)
\end{array}$$

The first thing to note is that our conclusion is a universal claim. So, let's try justifying it using an application of ∀I:

$$\begin{array}{r|ll}
1 & \forall x\big(Px \to \forall y\,(Ry \to Fxy)\big) & \\ \cline{2-2}
 & \ldots & \\
? & \forall y\big((Pa \land Ry) \to Fay\big) & \\
 & \forall x\, \forall y\big((Px \land Ry) \to Fxy\big) & \forall\text{I ?}
\end{array}$$

Working backwards, we have another universal claim to justify. So, Let's use another application of ∀ I:

1	$\forall x\big(Px \rightarrow \forall y\,(Ry \rightarrow Fxy)\big)$	
	$\ldots$	
??	$\big((Pa \wedge Rb) \rightarrow Fab\big)$	
?	$\forall y\big((Pa \wedge Ry) \rightarrow Fay\big)$	∀I ??
	$\forall x\,\forall y\big((Px \wedge Ry) \rightarrow Fxy\big)$	∀I ?

Again, working backwards, we now have a conditional claim to justify. So, we can try using →I:

1	$\forall x\big(Px \rightarrow \forall y\,(Ry \rightarrow Fxy)\big)$	
2	$\mid\; (Pa \wedge Rb)$	
	$\mid\; \ldots$	
???	$\mid\; Fab$	
??	$\big((Pa \wedge Rb) \rightarrow Fab\big)$	→I 2–???
?	$\forall y\big((Pa \wedge Ry) \rightarrow Fay\big)$	∀I ??
	$\forall x\,\forall y\big((Px \wedge Ry) \rightarrow Fxy\big)$	∀I ?

At this point, notice that our premise is a universal claim. So, we can try using ∀ E on it:

1	$\forall x\big(Px \rightarrow \forall y\,(Ry \rightarrow Fxy)\big)$	
2	$\mid\; (Pa \wedge Rb)$	
3	$\mid\; \big(Pa \rightarrow \forall y\,(Ry \rightarrow Fay)\big)$	∀E 1
	$\mid\; \ldots$	
???	$\mid\; Fab$	
??	$\big((Pa \wedge Rb) \rightarrow Fab\big)$	→I 2–???
?	$\forall y\big((Pa \wedge Ry) \rightarrow Fay\big)$	∀I ??
	$\forall x\,\forall y\big((Px \wedge Ry) \rightarrow Fxy\big)$	∀I ?

Here, we can easily detach the consequent of the conditional on 3 using *modus ponens*:

$$\begin{array}{lll}
1 & \vert\ \underline{\forall x\big(Px \to \forall y\,(Ry \to Fxy)\big)} & \\
2 & \vert\quad \vert\ \underline{(Pa \land Rb)} & \\
3 & \vert\quad \vert\ \big(Pa \to \forall y\,(Ry \to Fay)\big) & \forall\text{E } 1 \\
4 & \vert\quad \vert\ Pa & \land\text{E } 2 \\
5 & \vert\quad \vert\ \forall y\,(Ry \to Fay) & \to\text{E } 3, 4 \\
 & \vert\quad \vert\ \ldots & \\
??? & \vert\quad \vert\ Fab & \\
?? & \vert\ \big((Pa \land Rb) \to Fab\big) & \to\text{I } 2\text{–}??? \\
? & \vert\ \forall y\big((Pa \land Ry) \to Fay\big) & \forall\text{I } ?? \\
 & \vert\ \forall x\,\forall y\big((Px \land Ry) \to Fxy\big) & \forall\text{I } ?
\end{array}$$

Now notice that we have another universal claim at line 5. So, we can apply another instance of ∀ E on it to get $(Rb \to Fab)$, at which point we can use another instance of *modus ponens* to complete our proof:

$$\begin{array}{lll}
1 & \vert\ \underline{\forall x\big(Px \to \forall y\,(Ry \to Fxy)\big)} & \\
2 & \vert\quad \vert\ \underline{(Pa \land Rb)} & \\
3 & \vert\quad \vert\ \big(Pa \to \forall y\,(Ry \to Fay)\big) & \forall\text{E } 1 \\
4 & \vert\quad \vert\ Pa & \land\text{E } 2 \\
5 & \vert\quad \vert\ \forall y\,(Ry \to Fay) & \to\text{E } 3, 4 \\
6 & \vert\quad \vert\ (Rb \to Fab) & \forall\text{E } 5 \\
7 & \vert\quad \vert\ Rb & \land\text{E } 2 \\
8 & \vert\quad \vert\ Fab & \to\text{E } 6, 7 \\
9 & \vert\ \big((Pa \land Rb) \to Fab\big) & \to\text{I } 2\text{–}8 \\
10 & \vert\ \forall y\big((Pa \land Ry) \to Fay\big) & \forall\text{I } 9 \\
11 & \vert\ \forall x\,\forall y\big((Px \land Ry) \to Fxy\big) & \forall\text{I } 10
\end{array}$$

Let's take a look at a final proof. Suppose our UD consists in children in a kindergarten class. And suppose further that there is a girl that every boy likes. Surely, it follows from this that every boy likes some girl. In order to prove this, we first need to translate our premise and conclusion:

1	$\exists x\big(Gx \wedge \forall y\,(By \to Lyx)\big)$	
	$\ldots$	
	$\forall x\big(Bx \to \exists y\,(Gy \wedge Lxy)\big)$	

Since the conclusion is a universal claim, we know that we could justify it using an application of ∀ I:

1	$\exists x\big(Gx \wedge \forall y\,(By \to Lyx)\big)$	
	$\ldots$	
?	$\big(Ba \to \exists y\,(Gy \wedge Lay)\big)$	
	$\forall x\big(Bx \to \exists y\,(Gy \wedge Lxy)\big)$	∀I ?

Now we're tasked with proving a conditional. This we can do using an application of →I:

1	$\exists x\big(Gx \wedge \forall y\,(By \to Lyx)\big)$	
2	$\vert\ Ba$	
	$\vert\ \ldots$	
??	$\vert\ \exists y\,(Gy \wedge Lay)$	
?	$\big(Ba \to \exists y\,(Gy \wedge Lay)\big)$	→I 2–??
	$\forall x\big(Bx \to \exists y\,(Gy \wedge Lxy)\big)$	∀I ?

Working backwards, we now need to establish an existential claim. We can try to justify it using an application of ∃ E on line 1:

1	$\exists x\big(Gx \wedge \forall y\,(By \to Lyx)\big)$	
2	$\vert\ Ba$	
3	$\vert\ \vert\ \big(Gb \wedge \forall y\,(By \to Lyb)\big)$	
	$\vert\ \vert\ \ldots$	
???	$\vert\ \vert\ \exists y\,(Gy \wedge Lay)$	
??	$\vert\ \exists y\,(Gy \wedge Lay)$	∃E 1, 3–???
?	$\big(Ba \to \exists y\,(Gy \wedge Lay)\big)$	→I 2–??
	$\forall x\big(Bx \to \exists y\,(Gy \wedge Lxy)\big)$	∀I ?

At this point, we can eliminate both conjuncts at line 3, and use ∀E on the second conjunct:

1	$\exists x\big(Gx \wedge \forall y\,(By \to Lyx)\big)$	
2	Ba	
3	$\big(Gb \wedge \forall y\,(By \to Lyb)\big)$	
4	Gb	∧E 3
5	$\forall y\,(By \to Lyb)$	∧E 3
6	$(Ba \to Lab)$	∀E 5
	$\dots$	
???	$\exists y\,(Gy \wedge Lay)$	
??	$\exists y\,(Gy \wedge Lay)$	∃E 1, 3–???
?	$\big(Ba \to \exists y\,(Gy \wedge Lay)\big)$	→I 2–??
	$\forall x\big(Bx \to \exists y\,(Gy \wedge Lxy)\big)$	∀I ?

But now, we're almost done the proof. We can use *Bab* via *modus ponens* and use ∧I to get ($Gb \wedge Lab$). Finally, we can use an application of ∃I to get our existential claim needed to complete the proof:

1	$\exists x\big(Gx \wedge \forall y\,(By \to Lyx)\big)$	
2	Ba	
3	$\big(Gb \wedge \forall y\,(By \to Lyb)\big)$	
4	Gb	∧E 3
5	$\forall y\,(By \to Lyb)$	∧E 3
6	$(Ba \to Lab)$	∀E 5
7	Lab	→E 2, 6
8	$(Gb \wedge Lab)$	∧I 4, 7
9	$\exists y\,(Gy \wedge Lay)$	∃I 8
10	$\exists y\,(Gy \wedge Lay)$	∃E 1, 3–9
11	$\big(Ba \to \exists y\,(Gy \wedge Lay)\big)$	→I 2–10
12	$\forall x\big(Bx \to \exists y\,(Gy \wedge Lxy)\big)$	∀I 11

Practice exercises

Part A Translate the following English sentences into FOL using this symbolization key: (Hint: for some of these, you will need the inequality symbol.)

UD: Chess pieces on a chess board
Bx: x is black
Kx: x is a knight
Px: x is a pawn
Rx: x is a rook
Wx: x is white
Fxy: x is in front of y
Cxy: x is in the same column as y
Lxy: x is left of y
Oxy: x is in the same row as y

1. All of the black pieces are in front of all the white pieces.
2. Some rook is to the left of a knight.
3. All white pawns are in the same column.
4. Not all black pawns are in the same column.
5. Every rook is in a different row than every other rook. (Hint: you will need to use the inequality symbol to express this.)
6. Every pawn is in a different column than every other pawn.
7. Different knights are in the same column.
8. There are no different rooks in the same row.

Part B Using the symbolization key from part A, translate the following FOL sentences into English.

1. $\forall x \big(Px \to \exists y\,(Ry \land Lxy)\big)$
2. $\exists x \big(Px \land \forall y\,((Wy \land Ry) \to \neg Cxy)\big)$
3. $\forall x \big((Bx \land Kx) \to \neg\exists z\,(Wz \land (Pz \land Fzx))\big)$
4. $\exists x \big(Rx \land \neg\forall y\,(Py \to Oxy)\big)$
5. $\exists x \big(Rx \land \exists y\,(Ry \land x \neq y)\big)$
6. $\exists x\,\exists y\, x \neq y$
7. $\exists x\,\exists y\, Cxy \to \forall x\,\forall y\, Oxy$
8. $\forall x\,\forall y \big(((Rx \land Py) \land Fxy) \to \exists z\,(Kz \land Cxz)\big)$

Part C Using the following symbolization key, translate the following FOL sentences into English.

UD: Animals
Bx: x is brown
Dx: x is a dog
Fx: x is a frog
Gx: x is green
Mx: x is a mouse
Sxy: x is smaller than y
Ixy: x is bigger than y
m: Mighty Mouse

1. Every green frog is smaller than a brown dog.
2. Some frog is bigger than every mouse.
3. Nothing is bigger than everything.
4. Every frog bigger than every mouse is green.
5. Nothing smaller than a frog is bigger than a dog.
6. Some dog is smaller than some mouse.
7. No mouse is bigger than every dog.
8. Mighty Mouse is bigger than any other mouse.

Part D Provide formal proofs for the following valid inferences.

1. $\exists y\, \forall x\, Lyx$
   $\therefore\ \forall x\, \exists y\, Lyx$

2. $\exists x\, (Dx \wedge \exists y\, Rxy), \quad \forall x\, (\exists y\, Rxy \rightarrow Cx)$
   $\therefore\ \exists x\, Cx$

3. $\forall x\, \big(Px \rightarrow \exists y\, (Ry \wedge Lxy)\big), \quad \exists x\, Px$
   $\therefore\ \exists x\, Rx$

4. $\forall x\, \big(Px \rightarrow \forall y\, (Qy \rightarrow Ryx)\big), \quad \forall x\, \forall y\, (Rxy \rightarrow Txy)$
   $\therefore\ \forall x\, \big(Px \rightarrow \forall y\, (Qy \rightarrow Tyx)\big)$

5. $\exists x\, \big(Dx \wedge \neg\forall y\, (Cy \rightarrow Bxy)\big)$
   $\therefore\ \neg\forall x\, \big(Dx \rightarrow \forall y\, (Cy \rightarrow Bxy)\big)$

Chapter 11

Numerical quantification

In this final chapter, we look at how to express numerical statements in FOL, sentences that explicitly reference the numbers 1, 2, 3, and so on. We end by looking at some proofs that have numerical sentences as premises or conclusions.

11.1 Numerical statements

We've seen in previous chapters that we can express universal and existential statements in FOL using $\forall$ and $\exists$ respectively. One good thing about having a first-order language that contains identity (i.e., =) is that it allows us to say how many things there are of a particular thing. For example, consider these sentences:

1. There is at least one apple on the table.
2. There are at least two apples on the table.
3. There are at least three apples on the table.

Now, consider the following symbolization key:

UD: Everything
***Ax*:** x is an apple
***Tx*:** x is on the table

Sentence 1 doesn't require identity. It can be translated as $\exists x\,(Ax \wedge Tx)$: There is some apple on the table—perhaps many, but at least one. It might be tempting to translate sentence 2 without identity. Yet consider the sentence $\exists x\,\exists y\,[(Ax \wedge Tx) \wedge (Ay \wedge Ty)]$. It means that there is some apple x that is on the table and some apple y that is on the table. Since nothing precludes x and y from picking out the same member of the UD, this would be true even if there were only one apple on the table. In order to make sure that there are two *different* apples, we need the identity predicate. Sentence 2 needs to say that the two apples on the table are not identical. So, it can be translated as $\exists x\,\exists y\,[(Ax \wedge Tx) \wedge (Ay \wedge Ty) \wedge x \neq y]$.

Sentence 3 requires talking about three different apples on the table. It can be translated as $\exists x\,\exists y\,\exists z\,[(Ax \wedge Tx) \wedge (Ay \wedge Ty) \wedge (Az \wedge Tz) \wedge x \neq y \wedge y \neq z \wedge x \neq z]$. (We've omitted some brackets in this statement, and some others in this chapter, for clarity.)

Continuing this way, we can translate 'There are at least four apples on the table', 'There are at least five apples on the table', and so on.

Now consider these sentences:

4. There is at most one apple on the table.
5. There are at most two apples on the table.

Sentence 4 can be paraphrased as 'It is not the case that there are at least two apples on the table'. This is just the negation of sentence 2:

$$\neg\exists x\,\exists y\,[(Ax \wedge Tx) \wedge (Ay \wedge Ty) \wedge x \neq y].$$

Sentence 4 can also be approached in another way. It means that any apples that there are on the table must be the selfsame apple, so it can be translated as $\forall x\,\forall y\,\big[\big((Ax \wedge Tx) \wedge (Ay \wedge Ty)\big) \rightarrow x = y\big]$. The two translations are logically equivalent, so both are correct.

In a similar way, sentence 5 can be translated in two equivalent ways. It can be paraphrased as 'It is not the case that there are three or more distinct apples on the table', so it can be translated as the negation of sentence 3. Using universal quantifiers, it can also be translated as

$$\forall x\,\forall y\,\forall z\,\big[\big((Ax \wedge Tx) \wedge (Ay \wedge Ty) \wedge (Az \wedge Tz)\big) \rightarrow (x = y \vee y = z \vee x = z)\big].$$

We can also translate statements of equality which say exactly how many things there are. For example:

6. There is exactly one apple on the table.
7. There are exactly two apples on the table.

Sentence 7 can be paraphrased as 'There are at least two apples on the table, and there are at most two apples on the table'. This is just the conjunction of sentences 2 and 5:

$$\exists x\, \exists y\, [(Ax \wedge Tx) \wedge (Ay \wedge Ty) \wedge x \neq y]$$
$$\wedge$$
$$\forall x\, \forall y\, \forall z \Big[\big((Ax \wedge Tx) \wedge (Ay \wedge Ty) \wedge (Az \wedge Tz)\big) \rightarrow (x = y \vee y = z \vee x = z)\Big].$$

Admittedly, this translation is rather clunky and difficult to read. Luckily, we can approach sentence 7 in another way. To say that there are exactly two apples on the table is to say that there are at least two apples on the table, and any apple on the table is one of those apples. So, we can translate sentence 7 in the following way:

$$\exists x\, \exists y \Big[\big((Ax \wedge Tx) \wedge (Ay \wedge Ty) \wedge x \neq y\big) \wedge \forall z \big((Az \wedge Tz) \rightarrow (z = x \vee z = y)\big)\Big].$$

Similarly, sentence 6 can be paraphrased as ‘There is at least one apple on the table, and any apple on the table is that apple’. So, we can translate sentence 6 in the following way:

$$\exists x \Big[(Ax \wedge Tx) \wedge \forall y \big((Ay \wedge Ty) \rightarrow y = x\big)\Big].$$

11.2 Definite descriptions

In his article ‘On Denoting’ in 1905, Bertrand Russell asked how we should understand this sentence:

8. The present king of France is bald.

The phrase ‘the present king of France’ is supposed to pick out an individual by means of a definite description. However, there was no king of France in 1905 and there is none now. Since the description is a non-referring term, we cannot just define a constant (e.g., ‘*k*’) to mean ‘the present king of France’, take ‘*Bx*’ to mean ‘*x* is bald’, and translate the sentence as *Bk*.

Russell’s idea was that sentences that contain definite descriptions have a different logical structure than sentences that contain proper names, even though they are superficially similar. What do we mean when we use an unproblematic, referring description like ‘the highest peak in Washington state’? We mean that there is such a peak because we could not talk about it otherwise. We also mean that it is the only such peak. If there were another peak in Washington state of exactly the same height as Mount Rainier, then Mount Rainier would not be *the* highest peak.

According to this analysis, sentence 8 is saying three things. First, it makes an *existence* claim: There is some present king of France. Second, it makes a *uniqueness* claim: This guy is the only present king of France. Third, it makes a claim of *predication*: This guy is bald.

In order to symbolize definite descriptions in this way, we need the identity predicate. Without it, we could not translate the uniqueness claim which (according to Russell) is implicit in the definite description.

Let the UD be people actually living, let Fx mean 'x is the present king of France', and let Bx mean 'x is bald'. Sentence 8 can be translated as:

$$\exists x \Big[Fx \wedge \neg \exists y \, (Fy \wedge x \neq y) \wedge Bx \Big].$$

This says that there is some guy who is the present king of France, he is the only present kind of France, and he is bald.

Another way to translate sentence 8 gets its cue from the way we translated sentence 6. Sentence 8 can be paraphrased as 'There is at least one present king of France, and any present king of France is that guy, and that guy is bald':

$$\exists x \Big[Fx \wedge \forall y \, (Fy \rightarrow y = x) \wedge Bx \Big].$$

Understood in these ways, sentence 8 is meaningful but false. It say that this guy exists, but he does not.

The problem of non-referring terms is most vexing when we try to translate negations. So, consider this sentence:

9. The present king of France is not bald.

According to Russell, this sentence is ambiguous in English. It could mean either of two things:

9a. It is not the case that the present king of France is bald.
9b. The present king of France is non-bald.

Or in symbols:

9a. $\neg \exists x \Big[Fx \wedge \forall y \, (Fy \rightarrow y = x) \wedge Bx \Big]$
9b. $\exists x \Big[Fx \wedge \forall y \, (Fy \rightarrow y = x) \wedge \neg Bx \Big]$

Both possible translations add a negation to sentence 8, but they put the negation in different places. Sentence 9a is true; sentence 9b is false.

For a more detailed discussion of Russell's theory of definite descriptions, including objections to it, see Peter Ludlow's entry 'descriptions' in The Stanford Encyclopedia of Philosophy: Summer 2005, edited by Edward N. Zalta, `http://plato.stanford.edu/archives/sum2005/entries/descriptions/`.

11.3 Formal proofs using numerical quantification

As with proofs using multiple quantifiers, proofs using numerical quantification can be rather long and tricky. But if we use our rules correctly, we should have no problem.

Consider again sentence 4 above. To make our task more manageable, let us limit the UD to be *things on the table*. The following translations of sentence 4 are logically equivalent:

4a. $\neg\exists x\,\exists y\,[Ax \wedge Ay \wedge x \neq y]$
4b. $\forall x\,\forall y\,\big[(Ax \wedge Ay) \rightarrow x = y\big]$

This means that we can derive 4a from 4b and vice versa. Let's show that we can derive 4b from 4a:

1	$\neg\exists x\,\exists y\,[Ax \wedge Ay \wedge x \neq y]$
	$\dots$
	$\forall x\,\forall y\,\big[(Ax \wedge Ay) \rightarrow x = y\big]$

The first thing to note is that we can get our conclusion using two applications of ∀I:

1	$\neg\exists x\,\exists y\,[Ax \wedge Ay \wedge x \neq y]$	
	$\dots$	
??	$\big[(Ab \wedge Ac) \rightarrow b = c\big]$	
?	$\forall y\,\big[(Ab \wedge Ay) \rightarrow b = y\big]$	∀I ??
	$\forall x\,\forall y\,\big[(Ax \wedge Ay) \rightarrow x = y\big]$	∀I ?

Our next task is to establish a conditional claim using →I:

1	$\neg\exists x\,\exists y\,[Ax \land Ay \land x \neq y]$	
2	$\quad (Ab \land Ac)$	
	$\quad \ldots$	
???	$\quad b = c$	
??	$\big[(Ab \land Ac) \to b = c\big]$	→I 2–???
?	$\forall y\,\big[(Ab \land Ay) \to b = y\big]$	∀I ??
	$\forall x\,\forall y\,\big[(Ax \land Ay) \to x = y\big]$	∀I ?

We now need to establish $b = c$. We can do this via *reductio* using ¬E:

1	$\neg\exists x\,\exists y\,[Ax \land Ay \land x \neq y]$	
2	$\quad (Ab \land Ac)$	
3	$\quad\quad b \neq c$	
	$\quad\quad \ldots$	
????		
???	$\quad b = c$	¬E 3–????
??	$\big[(Ab \land Ac) \to b = c\big]$	→I 2–???
?	$\forall y\,\big[(Ab \land Ay) \to b = y\big]$	∀I ??
	$\forall x\,\forall y\,\big[(Ax \land Ay) \to x = y\big]$	∀I ?

Now, in order to properly use the ¬E rule, we need to end our subproof with a contradiction—i.e., a sentence and its negation. But we can see that lines 2 and 3 state that *a* and *b* are two different apples on the table, which contradicts our premise. So, with an application of ∧I, and two applications of ∃I, we can finish our proof:

1	$\neg\exists x\,\exists y\,[Ax \wedge Ay \wedge x \neq y]$	
2	$\mid\ (Ab \wedge Ac)$	
3	$\mid\ \mid\ b \neq c$	
4	$\mid\ \mid\ \big(Ab \wedge Ac \wedge b \neq c\big)$	∧I 2, 3
5	$\mid\ \mid\ \exists y\big(Ab \wedge Ay \wedge b \neq y\big)$	∃I 4
6	$\mid\ \mid\ \exists x\,\exists y\big(Ax \wedge Ay \wedge x \neq y\big)$	∃I 5
7	$\mid\ \mid\ \neg\exists x\,\exists y\big(Ax \wedge Ay \wedge x \neq y\big)$	R 1
8	$\mid\ b = c$	¬I 3–7
9	$\big[(Ab \wedge Ac) \rightarrow b = c\big]$	→I 2–8
10	$\forall y\big[(Ab \wedge Ay) \rightarrow b = y\big]$	∀I 9
11	$\forall x\,\forall y\big[(Ax \wedge Ay) \rightarrow x = y\big]$	∀I 10

In order to prove that 4a and 4b are logically equivalent, we would now need to show that we can also derive 4a from 4b. This proof is a bit tricker, requiring various nested subproofs. We will leave it as a practice exercise (see Practice Exercises PART C, exercise d below).

Let's take a look at a final proof, where we show that there exists at least two *P*s logically entails that it is not the case that there is at most one *P*.

1	$\exists x\,\exists y\,[Px \wedge Py \wedge x \neq y]$	
	$\ldots$	
	$\neg\forall x\,\forall y\big[(Px \wedge Py) \rightarrow x = y\big]$	

Since our conclusion is a negation, we could justify it via *reductio*. But since we've got an existential statement as a premise, we could try to justify our conclusion using an application of ∃E. Let's do that:

1	$\exists x \exists y\,[Px \land Py \land x \neq y]$	
2	$\vert\quad \exists y\,[Pa \land Py \land a \neq y]$	
	$\vert\quad \dots$	
?	$\vert\quad \neg\forall x \forall y\,\big[(Px \land Py) \to x = y\big]$	
	$\neg\forall x \forall y\,\big[(Px \land Py) \to x = y\big]$	∃E 1, 2–?

Let's now use another application of ∃E to eliminate the statement on line 2:

1	$\exists x \exists y\,[Px \land Py \land x \neq y]$	
2	$\vert\quad \exists y\,[Pa \land Py \land a \neq y]$	
3	$\vert\quad \vert\quad (Pa \land Pb \land a \neq b)$	
	$\vert\quad \vert\quad \dots$	
??	$\vert\quad \vert\quad \neg\forall x \forall y\,\big[(Px \land Py) \to x = y\big]$	
?	$\vert\quad \neg\forall x \forall y\,\big[(Px \land Py) \to x = y\big]$	∃E 2, 3–??
	$\neg\forall x \forall y\,\big[(Px \land Py) \to x = y\big]$	∃E 1, 2–?

Now we're tasked with justifying a negation. Let's try here using an application of ¬I:

1	$\exists x \exists y\,[Px \land Py \land x \neq y]$	
2	$\vert\quad \exists y\,[Pa \land Py \land a \neq y]$	
3	$\vert\quad \vert\quad (Pa \land Pb \land a \neq b)$	
4	$\vert\quad \vert\quad \vert\quad \forall x \forall y\,\big[(Px \land Py) \to x = y\big]$	
	$\vert\quad \vert\quad \vert\quad \dots$	
???	$\vert\quad \vert\quad \vert$	
??	$\vert\quad \vert\quad \neg\forall x \forall y\,\big[(Px \land Py) \to x = y\big]$	¬I 4–???
?	$\vert\quad \neg\forall x \forall y\,\big[(Px \land Py) \to x = y\big]$	∃E 2, 3–??
	$\neg\forall x \forall y\,\big[(Px \land Py) \to x = y\big]$	∃E 1, 2–?

We know that we must end our subproof with a contradiction—i.e., some sentence and its negation. Let's now do two applications of ∀E (the first on line 4) to see whether we can unearth that contradiction:

1	$\mid\ \exists x\, \exists y\, [Px \land Py \land x \neq y]$	
2	$\mid\ \mid\ \exists y\, [Pa \land Py \land a \neq y]$	
3	$\mid\ \mid\ \mid\ (Pa \land Pb \land a \neq b)$	
4	$\mid\ \mid\ \mid\ \mid\ \forall x\, \forall y \big[(Px \land Py) \to x = y\big]$	
5	$\mid\ \mid\ \mid\ \mid\ \forall y \big[(Pa \land Py) \to a = y\big]$	∀E 4
6	$\mid\ \mid\ \mid\ \mid\ \big[(Pa \land Pb) \to a = b\big]$	∀E 5
	$\mid\ \mid\ \mid\ \mid\ \ldots$	
???	$\mid\ \mid\ \mid\ \mid$	
??	$\mid\ \mid\ \mid\ \neg\forall x\, \forall y \big[(Px \land Py) \to x = y\big]$	¬I 4–???
?	$\mid\ \mid\ \neg\forall x\, \forall y \big[(Px \land Py) \to x = y\big]$	∃E 2, 3–??
	$\mid\ \neg\forall x\, \forall y \big[(Px \land Py) \to x = y\big]$	∃E 1, 2–?

We can now establish our contradiction, and thus finish our proof, with a few applications of ∧E, one application of ∧I, and an application of →E:

1	$\mid\ \exists x\, \exists y\, [Px \land Py \land x \neq y]$	
2	$\mid\ \mid\ \exists y\, [Pa \land Py \land a \neq y]$	
3	$\mid\ \mid\ \mid\ (Pa \land Pb \land a \neq b)$	
4	$\mid\ \mid\ \mid\ \mid\ \forall x\, \forall y \big[(Px \land Py) \to x = y\big]$	
5	$\mid\ \mid\ \mid\ \mid\ \forall y \big[(Pa \land Py) \to a = y\big]$	∀E 4
6	$\mid\ \mid\ \mid\ \mid\ \big[(Pa \land Pb) \to a = b\big]$	∀E 5
7	$\mid\ \mid\ \mid\ \mid\ Pa$	∧E 3
8	$\mid\ \mid\ \mid\ \mid\ Pb$	∧E 3
9	$\mid\ \mid\ \mid\ \mid\ (Pa \land Pb)$	∧I 7, 8
10	$\mid\ \mid\ \mid\ \mid\ a = b$	→E 6, 9
11	$\mid\ \mid\ \mid\ \mid\ a \neq b$	∧E 3
12	$\mid\ \mid\ \mid\ \neg\forall x\, \forall y \big[(Px \land Py) \to x = y\big]$	¬I 4–11
13	$\mid\ \mid\ \neg\forall x\, \forall y \big[(Px \land Py) \to x = y\big]$	∃E 2, 3–12
14	$\mid\ \neg\forall x\, \forall y \big[(Px \land Py) \to x = y\big]$	∃E 1, 2–13

Practice exercises

Part A Write a symbolization key and symbolize the following statements. (Hint: Use ‘animals at the zoo’ as your UD.)

1. There’s a rhino at the zoo.
2. Rodney is the only rhino at the zoo.
3. There is only one rhino at the zoo.
4. There are at least two lions at the zoo.
5. There are at most two lions at the zoo.
6. There are exactly two lions at the zoo.
7. A lion is sleeping.
8. The lion is sleeping.

Part B Using the following symbolization key, translate the following statements into English:

UD: The people at a certain party
***Dx*:** x is dancing
***Lxy*:** x loves y
***a*:** Ashni
***b*:** Ben

1. $\big(Lab \land \forall x\,(Lxb \rightarrow x = a)\big)$
2. $\forall z\,(Lzb \leftrightarrow z = b)$
3. $\exists x\,\exists y\,(Dx \land Dy \land x \neq y)$
4. $\exists x\,\big(Lxa \land \forall y\,(Lya \rightarrow y = x) \land \neg Dx\big)$
5. $\exists x\,\exists y\,\big(x \neq y \land \forall z\,(z = x \lor z = y)\big)$

Part C Show that the following are valid, using natural deduction proofs:

1. $\exists x\,\exists y\,(Px \land Py \land x \neq y)$
$\therefore\ \exists x\,Px$

2. **(Challenging)**
$\exists x\,\big[Px \land \forall y\,(Py \rightarrow y = x) \land Qx\big]$
$\therefore\ \forall x\,\forall y\,\big[(Px \land Py) \rightarrow x = y\big]$

3. **(Challenging)**

$\forall x\, \forall y \big[\neg(Tx \land Ty) \lor x = y\big]$

$\therefore\ \forall x\, \neg\exists y \big[(Tx \land Ty) \land x \neq y\big]$

4. **(Challenging)**

$\forall x\, \forall y \big[(Ax \land Ay) \to x = y\big]$

$\therefore\ \neg\exists x\, \exists y \big(Ax \land Ay \land x \neq y\big)$

5. **(Challenging)**

$\exists x \big[Tx \land \forall y\, (Ty \to y = x)\big]$

$\therefore\ \neg\exists x\, \exists y \big(Tx \land Ty \land x \neq y\big)$

Chapter 12

Quick reference

Characteristic Truth Tables

$\mathcal{P}$	$\neg\mathcal{P}$
T	F
F	T

$\mathcal{P}$	$\mathcal{Q}$	$(\mathcal{P} \wedge \mathcal{Q})$	$(\mathcal{P} \vee \mathcal{Q})$	$(\mathcal{P} \to \mathcal{Q})$	$(\mathcal{P} \leftrightarrow \mathcal{Q})$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Rules of Proof

Reiteration (R)

j	$\mathcal{P}$	
k	$\mathcal{P}$	R j

Conjunction Introduction ($\wedge$I)

j	$\mathcal{P}$	
k	$\mathcal{Q}$	
l	$(\mathcal{P} \wedge \mathcal{Q})$	$\wedge$I j, k

Conjunction Elimination (∧E)

j	$(\mathcal{P} \wedge \mathcal{Q})$		j	$(\mathcal{P} \wedge \mathcal{Q})$	
k	$\mathcal{P}$	∧E j	k	$\mathcal{Q}$	∧E j

Disjunction Introduction (∨I)

j	$\mathcal{P}$		j	$\mathcal{Q}$	
k	$(\mathcal{P} \vee \mathcal{Q})$	∨I j	k	$(\mathcal{P} \vee \mathcal{Q})$	∨I j

Disjunction Elimination (∨E)

j	$(\mathcal{P} \vee \mathcal{Q})$		j	$(\mathcal{P} \vee \mathcal{Q})$	
k	$\neg\mathcal{Q}$		k	$\neg\mathcal{P}$	
l	$\mathcal{P}$	∨E j, k	l	$\mathcal{Q}$	∨E j, k

Conditional Introduction (→I)

j	$\quad\vert\ \underline{\mathcal{P}}$	
k	$\quad\vert\ \mathcal{Q}$	
l	$(\mathcal{P} \rightarrow \mathcal{Q})$	→I j–k

Conditional Elimination (→E)

j	$(\mathcal{P} \rightarrow \mathcal{Q})$	
k	$\mathcal{P}$	
l	$\mathcal{Q}$	→E j, k

BICONDITIONAL INTRODUCTION (↔I)

h	$\begin{array}{\|l} \mathcal{P} \\ \hline \end{array}$	
i	$\begin{array}{\|l} \mathcal{Q} \end{array}$	
j	$\begin{array}{\|l} \mathcal{Q} \\ \hline \end{array}$	
k	$\begin{array}{\|l} \mathcal{P} \end{array}$	
l	$(\mathcal{P} \leftrightarrow \mathcal{Q})$	↔I h–i, j–k

BICONDITIONAL ELIMINATION (↔E)

j	$(\mathcal{P} \leftrightarrow \mathcal{Q})$	
k	$\mathcal{P}$	
l	$\mathcal{Q}$	↔E j, k

j	$(\mathcal{P} \leftrightarrow \mathcal{Q})$	
k	$\mathcal{Q}$	
l	$\mathcal{P}$	↔E j, k

Identity Rules

IDENTITY INTRODUCTION (=I)

j	$n = n$	=I 1

IDENTITY ELIMINATION (=E)

j	$\mathcal{P}n$	
k	$n = m$	
l	$\mathcal{P}m$	=E j, k

Quantifier Rules

UNIVERSAL INTRODUCTION (∀I)

j	$\mathcal{A}c^*$	
k	$\forall\mathcal{x}\,\mathcal{A}\mathcal{x}$	∀I j

* The constant c must not occur in any undischarged assumption.

UNIVERSAL ELIMINATION (∀E)

j	$\forall\mathcal{x}\,\mathcal{A}\mathcal{x}$	
k	$\mathcal{A}c$	∀E j

Existential Introduction (∃I)

$$\begin{array}{r|ll}
j & \mathcal{A}c & \\
k & \exists\chi\,\mathcal{A}\chi & \exists\text{I } j
\end{array}$$

Existential Elimination (∃E)

$$\begin{array}{r|ll}
j & \exists\chi\,\mathcal{A}\chi & \\
\begin{array}{r} k \\ l \end{array} & \quad\begin{array}{|l} \mathcal{A}c^{*} \\ \hline \mathcal{B} \end{array} & \\
m & \mathcal{B} & \exists\text{E } j, k\text{–}l
\end{array}$$

* The constant c must not appear in $\exists\chi\,\mathcal{A}\chi$, in $\mathcal{B}$, or in any undischarged assumption.

Part III

Solutions

Chapter 13

Solutions to exercises

Many of the exercises may be answered correctly in different ways. Where that is the case, the solution here represents one possible correct answer.

13.1 Chapter 1 Solutions

CHAPTER 1 PART A

1. *Statement:* England is smaller than China.
2. *Statement:* Greenland is south of Jerusalem.
3. *Not a statement:* Is New Jersey east of Wisconsin?
4. *Statement:* The atomic number of helium is 2.
5. *Statement:* The atomic number of helium is π.
6. *Statement:* I hate overcooked noodles.
7. *Not a statement:* Blech! Overcooked noodles!
8. *Statement:* Overcooked noodles are disgusting.
9. *Not a statement:* Take your time.
10. *Statement:* This is the last question.

CHAPTER 1 PART B

1. *Contingent*: Caesar crossed the Rubicon.
2. *Contingent*: Someone once crossed the Rubicon.
3. *Contingent*: No one has ever crossed the Rubicon.
4. *Logical Truth*: If Caesar crossed the Rubicon, then someone has.
5. *Logical Falsehood*: Even though Caesar crossed the Rubicon, no one has ever crossed the Rubicon.
6. *Contingent*: If anyone has ever crossed the Rubicon, it was Caesar.

CHAPTER 1 PART C

1. consistent
2. inconsistent
3. consistent
4. consistent

CHAPTER 1 PART D

1. *Possible:* A valid inference that has one false premise and one true premise
    1. Ottawa is the national capital of Canada. (true)
    2. Ottawa is in Manitoba. (false)
    3. ∴ The national capital is in Manitoba. (false)
2. *Possible:* A valid inference that has a false conclusion
    1. Ottawa is the national capital of Canada. (true)
    2. Ottawa is in Manitoba. (false)
    3. ∴ The national capital is in Manitoba. (false)
3. *Possible:* A valid inference, the conclusion of which is a logical falsehood
    1. It is raining outside.
    2. It is not raining outside
    3. ∴ It is both raining and not raining outside.
4. *Not possible:* An invalid inference, the conclusion of which is a tautology
5. *Not possible:* A logical truth that is contingent
6. *Possible:* Two logically equivalent sentences, both of which are tautologies
    1. Either the Habs will make the playoffs this year or they won't.
    2. The Habs won't both make and not make the playoffs this year.
7. *Not possible:* Two logically equivalent sentences, one of which is a tautology and one of which is contingent
8. *Possible:* Two logically equivalent sentences that together are an inconsistent set
    1. It is neither raining nor not raining outside.
    2. It is both raining and not raining outside.
9. *Not possible:* A consistent set of sentences that contains a contradiction
10. *Possible:* An inconsistent set of sentences that contains a tautology
    1. It is either raining or not raining outside.
    2. It is raining outside.
    3. it is not raining outside.

Chapter 1 Part E The argument can be reconstructed as follows:

1. Ashni told me that she only sees Ben once or twice a year. (Premise)
2. Ashni and Ben live far apart. (From 1)
3. Ashni and Ben don't both live in New York. (From 2)
4. Ashni lives in New York. (Premise)
5. Ben doesn't live in New York. (From 3, 4)

The inference from (3) and (4) to (5) is valid; the other inferences are not.

13.2 Chapter 2 Solutions

CHAPTER 2 PART A

1. Bob is a man in a suit.
Mb
2. Bob is a man in a suit or he is not.
$(Mb \lor \neg Mb)$
3. Koko is either a gorilla or a chimpanzee.
$(Gk \lor Ck)$
4. Bob is neither a gorilla nor a chimpanzee.
$\neg(Gb \lor Cb)$
5. Flo is neither a gorilla nor a man in a suit, and nor a chimpanzee.
$\big(\neg(Gf \lor Mf) \land \neg Cf\big)$
6. Flo is either a gorilla or a chimpanzee, not a man in a suit.
$\big((Gf \lor Cf) \land \neg Mf\big)$

CHAPTER 2 PART B

1. Either Mister Ace or Mister Edge was murdered.
$(M_1a \lor M_1e)$
2. Mister Ace and Mister Edge weren't both murdered.
$\neg(M_1a \land M_1e)$
3. Either the cook did it, or the butler did it.
$\big((M_2ca \lor M_2ce) \lor (M_2ba \lor M_2be)\big)$
4. Either the butler did it, or the Duchess is lying.
$\big((M_2ba \lor M_2be) \lor Ld\big)$
5. Mister Edge was murdered and the cook did it.
$(M_e \land M_2ce)$
6. Either the murder weapon was a frying pan or the Duchess isn't lying.
$(Fw \lor \neg Ld)$
7. Either the Duchess is lying, or the culprit is either the cook or the butler.
$\Big[Ld \lor \big((M_2ca \lor M_2ce) \lor (M_2ba \lor M_2be)\big)\Big]$
8. Either the Duchess is lying, or both Mister Edge and Mister Ace were murdered.
$\big(Ld \lor (M_1e \land M_1a)\big)$
9. Neither the butler nor the cook did it.
$\neg\big((M_2ca \lor M_2ce) \lor (M_2ba \lor M_2be)\big)$
10. Although the murder weapon was a frying pan, either both Mister Edge and Mister Ace were murdered or the Duchess is lying.
$\Big[Fw \land \big((M_1e \land M_1a) \lor Ld\big)\Big]$
11. Of course the Duchess is lying!
Ld

CHAPTER 2 PART C

1. Ava and Harrison are both electricians.
   $(Ea \wedge Eh)$
2. Ava is a firefighter satisfied with her career.
   $(Fa \wedge Sa)$
3. Ava is either a firefighter or an electrician.
   $(Fa \vee Ea)$
4. Harrison is an unsatisfied electrician.
   $(Eh \wedge \neg Sh)$
5. Neither Ava nor Harrison is an electrician.
   $\neg(Ea \vee Eh)$
6. Both Ava and Harrison are electricians, but neither of them find it satisfying.
   $\big((Ea \wedge Eh) \wedge \neg(Sa \vee Sh)\big)$
7. Harrison is either an unsatisfied firefighter or a satisfied electrician.
   $\big((\neg Sh \wedge Fh) \vee (Sh \wedge Eh)\big)$
8. Harrison and Ava are both firefighters who are satisfied with their careers.
   $\big((Fh \wedge Fa) \wedge (Sh \wedge Sa)\big)$
9. Either Harrison and Ava are both firefighters or neither of them is a firefighter.
   $\big((Fh \wedge Fa) \vee \neg(Fh \vee Fa)\big)$

CHAPTER 2 PART D Here is how you can translate an *exclusive or* using only two connectives:
$\big(\neg(\neg P \wedge \neg Q) \wedge \neg(P \wedge Q)\big)$

13.3 Chapter 3 Solutions

CHAPTER 3 PART A

1. As the following truth table shows, this sentence is a contradiction.

Hab	$(\neg$	Hab	$\wedge$	$Hab)$
T	F		**F**	
F	T		**F**	

2. As the following truth table shows, this sentence is neither a tautology nor a contradiction. It is a tautologically contingent statement.

Pa	Hab	$(\neg$	Pa	$\vee$	$Hab)$
T	T	F		**T**	
T	F	F		**F**	
F	T	T		**T**	
F	F	T		**T**	

3. As the following truth table shows, this sentence is a tautology.

Rcd	$(\neg$	Rcd	$\vee$	$Rcd)$
T	F		**T**	
F	T		**T**	

4. As the following truth table shows, this sentence is neither a tautology nor a contradiction. It is a tautologically contingent statement.

Pa	Hab	$((Pa$	$\wedge$	$Hab)$	$\vee$	$(Hab$	$\wedge$	$Pa))$
T	T		T		**T**		T	
T	F		F		**F**		F	
F	T		F		**F**		F	
F	F		F		**F**		F	

5. As the following truth table shows, this sentence is a contradiction.

Pa	Hab	Tc	$[((Pa$	$\wedge$	$Hab)$	$\wedge$	$\neg$	$(Pa$	$\wedge$	$Hab))$	$\wedge$	$Tc]$
T	T	T		T		F	F		T		**F**	
T	T	F		T		F	F		T		**F**	
T	F	T		F		F	T		F		**F**	
T	F	F		F		F	T		F		**F**	
F	T	T		F		F	T		F		**F**	
F	T	F		F		F	T		F		**F**	
F	F	T		F		F	T		F		**F**	
F	F	F		F		F	T		F		**F**	

6. As the following truth table shows, this sentence is neither a tautology nor a contradiction. It is a tautologically contingent statement.

Pa	Hab	$\big[\big((Pa \wedge Hab)$	$\vee$	$(Pa \wedge$	$\neg Hab)\big)$	$\vee$	$\neg Hab\big]$
T	T	T	T	F	F	**T**	F
T	F	F	T	T	T	**T**	T
F	T	F	F	F	F	**F**	F
F	F	F	F	F	T	**T**	T

Chapter 3 Part B

1. The following joint truth table shows that the sentences are not tautologically equivalent:

Pa	Pa	$\neg Pa$
T	**T**	**F**
F	**F**	**T**

2. The following joint truth table shows that the sentences are tautologically equivalent:

Pa	Pa	$(Pa \vee Pa)$
T	T	T
F	F	F

3. The following joint truth table shows that the sentences are tautologically equivalent:

Pa	Hab	$\neg$	$(Pa \wedge Hab)$	$(\neg Pa$	$\vee$	$\neg Hab)$
T	T	**F**	T	F	**F**	F
T	F	**T**	F	F	**T**	T
F	T	**T**	F	T	**T**	F
F	F	**T**	F	T	**T**	T

4. The following joint truth table shows that the sentences are not tautologically equivalent:

Pa	Hab	Tc	$\big((Pa \vee Hab)$	$\wedge Tc\big)$	$\big(Pa \vee$	$(Hab \wedge Tc)\big)$
T	T	T	T	**T**	**T**	T
T	T	F	T	**F**	**T**	F
T	F	T	T	**T**	**T**	F
T	F	F	T	**F**	**T**	F
F	T	T	T	**T**	**T**	T
F	T	F	T	**F**	**F**	F
F	F	T	F	**F**	**F**	F
F	F	F	F	**F**	**F**	F

CHAPTER 3 PART C

1. The following joint truth table shows that the argument is tautologically valid:

Pa	Hab	Tc	$(Pa \wedge Hab)$	Tc	$(Hab \wedge Tc)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	F	T	T
F	T	F	F	F	F
F	F	T	F	T	F
F	F	F	F	F	F

2. The following joint truth table shows that the argument is not tautologically valid:

Pa	Hab	Tc	$(Pa \vee Hab)$	Tc	$(Hab \wedge Tc)$
T	T	T	T	T	T
T	T	F	T	F	F
T	**F**	**T**	**T**	**T**	**F** ★
T	F	F	T	F	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	F	F	F

3. The following joint truth table shows that the argument is not tautologically valid:

Pa	Hab	Tc	$(Pa \vee Hab)$	$(Hab \vee Tc)$	$\neg Pa$	$(Hab \wedge Tc)$
T	T	T	T	T	F	T
T	T	F	T	T	F	F
T	F	T	T	T	F	F
T	F	F	T	F	F	F
F	T	T	T	T	T	T
F	**T**	**F**	**T**	**T**	**T**	**F** ★
F	F	T	F	T	T	F
F	F	F	F	F	T	F

4. The following joint truth table shows that the argument is tautologically valid:

Pa	Hab	Tc	$(Pa \vee Hab)$	$(Hab \vee Tc)$	$\neg Hab$	$(Pa \wedge Tc)$
T	T	T	T	T	F	T
T	T	F	T	T	F	F
T	F	T	T	T	T	T
T	F	F	T	F	T	F
F	T	T	T	T	F	F
F	T	F	T	T	F	F
F	F	T	F	T	T	F
F	F	F	F	F	T	F

CHAPTER 3 PART D

Given the definitions of 'tautology' and 'contradiction', it follows that there cannot be any atomic tautology or contradiction.

13.4 Chapter 4 Solutions

Chapter 4 Part A

1. If Flo is a chimpanzee, then she is not a gorilla.
   $(Cf \to \neg Gf)$
2. If Koko is not a man in a suit, then she's either a chimpanzee or a gorilla.
   $\big(\neg Mk \to (Ck \lor Gk)\big)$
3. If Bob is a chimpanzee, then he is neither a gorilla nor a chimpanzee.
   $\big(Cb \to \neg(Gb \lor Cb)\big)$
4. Unless Bob is a man in a suit, he is either a chimpanzee or a gorilla.
   $\big(\neg Mb \to (Cb \lor Gb)\big)$

Chapter 4 Part B

1. If Mister Ace was murdered, then the cook did it.
   $(M_1a \to M_2ca)$
2. If Mister Edge was murdered, then the cook did not do it.
   $(M_1e \to \neg M_2ce)$
3. The cook did it only if the Duchess is lying.
   $(M_2ca \to Ld)$
4. If the murder weapon was a frying pan, then the culprit must have been the cook.
   $(Fw \to M_2ca)$
5. If the murder weapon was not a frying pan, then the culprit was either the cook or the butler.
   $\big(\neg Fw \to (M_2ca \lor M_2be)\big)$
6. Mister Ace was murdered if and only if Mister Edge was not murdered.
   $(M_1a \leftrightarrow \neg M_1e)$
7. The Duchess is lying, unless it was Mister Edge who was murdered.
   $(\neg M_1e \to Ld)$
8. If Mister Ace was murdered, he was murdered with a frying pan.
   $(M_1a \to Fw)$

Chapter 4 Part C

1. If Ava is a firefighter, then she is satisfied with her career.
   $(Fa \to Sa)$
2. Ava is a firefighter, unless she is an electrician.
   $(\neg Ea \to Fa)$
3. Harrison is satisfied only if he is firefighter.
   $(Sh \to Fh)$

4. If Ava is not an electrician, then neither is Harrison, but if she is, then he is too.
   $\big((\neg Ea \rightarrow \neg Eh) \wedge (Ea \rightarrow Eh)\big)$
5. Ava is satisfied with her career if and only if Harrison is not satisfied with his.
   $(Sa \leftrightarrow \neg Sh)$
6. If Harrison is both an electrician and a firefighter, then he must be satisfied with his work.
   $\big((Eh \wedge Fh) \rightarrow Sh\big)$
7. Harrison and Ava are both firefighters if and only if neither of them is an electrician.
   $\big((Fh \wedge Fa) \leftrightarrow \neg(Eh \vee Ea)\big)$

Chapter 4 Part D

Consider the following symbolization key:

R_1x: x has been broken
R_2xy: x broke y
Sx: x is a spy
Ux: x is in uproar
a: Alice
b: Bob
c: the code
g: the German embassy

The sentences can be translated thusly:

1. Alice and Bob are both spies.
   $(Sa \wedge Sb)$
2. If either Alice or Bob is a spy, then the code has been broken.
   $\big((Sa \vee Sb) \rightarrow R_1c\big)$
3. If neither Alice nor Bob is a spy, then the code remains unbroken.
   $\big(\neg(Sa \vee Sb) \rightarrow \neg R_1c\big)$
4. The German embassy will be in an uproar, unless someone has broken the code.
   $\big(\neg(R_2ac \wedge R_2bc) \rightarrow Ug\big)$
5. Either the code has been broken or it has not, but the German embassy will be in an uproar regardless.
   $\big((R_1c \vee \neg R_1c) \wedge Ug\big)$
6. Either Alice or Bob is a spy, but not both.
   $\neg(Sa \leftrightarrow Sb)$

CHAPTER 4 PART E

1. Consider the following symbolization key:

   Cx: x wakes up cranky
   Dx: x is distracted
   Px: x plays the piano in the morning
   o: Dorthy
   r: Roger

   The argument can be translated thusly:

   $(Po \rightarrow Cr), \quad (\neg Dr \rightarrow Po) \quad \therefore \quad (\neg Cr \rightarrow Do)$

2. Consider the following symbolization key:

   Px: The precipitation is x
   Cx: x is cold
   Sx: x is sad
   n: Neville
   o: snow
   r: rain

   The argument can be translated thusly:

   $(Pr \vee Po), \quad (Pr \rightarrow Sn), \quad (Po \rightarrow Cn) \quad \therefore \quad (Sn \vee Cn)$

3. Consider the following symbolization key:

   Cx: x is clean
   Nx: x is neat
   Rx: x remembered to do his chores
   o: Zoog
   t: things

   The argument can be translated thusly:

   $\big(Ro \rightarrow (Ct \wedge \neg Nt)\big), \quad \big(\neg Ro \rightarrow (Nt \wedge \neg Ct)\big) \quad \therefore \quad \neg(Nt \leftrightarrow Ct)$

CHAPTER 4 PART F

1. Valid: $(Pa \rightarrow Qa), \quad Pa \quad \therefore \quad Qa$

Pa	Qa	$(Pa \rightarrow Qa)$	Pa	Qa
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

2. Invalid: $(Pa \rightarrow Qa)$, Qa $\therefore$ Pa

Pa	Qa	$(Pa \rightarrow Qa)$	Qa	Pa
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F ★
F	F	T	F	F

3. Valid: $(Pa \rightarrow Qa)$, $\neg Qa$ $\therefore$ $\neg Pa$

Pa	Qa	$(Pa \rightarrow Qa)$	$\neg Qa$	$\neg Pa$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

4. Invalid: $(Pa \rightarrow Qa)$, $\neg Pa$ $\therefore$ $\neg Qa$

Pa	Qa	$(Pa \rightarrow Qa)$	$\neg Pa$	$\neg Qa$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F ★
F	F	T	T	T

5. Valid: $(Pa \leftrightarrow Qa)$ $\therefore$ $(Qa \leftrightarrow Pa)$

Pa	Qa	$(Pa \leftrightarrow Qa)$	$(Qa \leftrightarrow Pa)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

6. Valid: $(Pa \leftrightarrow Qa)$ $\therefore$ $(\neg Pa \leftrightarrow \neg Qa)$

Pa	Qa	$(Pa \leftrightarrow Qa)$	$(\neg Pa \leftrightarrow \neg Qa)$
T	T	T	F **T** F
T	F	F	F **F** T
F	T	F	T **F** F
F	F	T	T **T** T

7. Valid: $(Pa \leftrightarrow Qa)$ $\therefore$ $(Pa \vee \neg Qa)$

Pa	Qa	$(Pa \leftrightarrow Qa)$	$(Pa \vee \neg Qa)$
T	T	T	**T** F
T	F	F	**T** T
F	T	F	**F** F
F	F	T	**T** T

8. Valid: $(Pa \to Qa) \quad \therefore \quad (\neg Qa \to \neg Pa)$

Pa	Qa	$(Pa \to Qa)$	$(\neg\, Qa \to \neg\, Pa)$
T	T	T	F **T** F
T	F	F	T **F** F
F	T	T	F **T** T
F	F	T	T **T** T

9. Invalid: $(Pa \to Qa) \quad \therefore \quad (\neg Pa \to \neg Qa)$

Pa	Qa	$(Pa \to Qa)$	$(\neg\, Pa \to \neg\, Qa)$	
T	T	T	F **T** F	
T	F	F	F **T** T	
F	T	T	T **F** F	⋆
F	F	T	T **T** T	

10. Valid: $(Pa \to Qa), \quad (Qa \to Ra) \quad \therefore \quad (Pa \to Ra)$

Pa	Qa	Ra	$(Pa \to Qa)$	$(Pa \to Qa)$	$(Qa \to Ra)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

11. Valid: $\big(Pa \vee (Qa \to Pa)\big) \quad \therefore \quad (\neg Pa \to \neg Qa)$

Pa	Qa	$\big(Pa \vee (Qa \to Pa)\big)$	$(\neg\, Pa \to \neg\, Qa)$
T	T	**T** T	F **T** F
T	F	**T** T	F **T** T
F	T	**F** F	T **F** F
F	F	**T** T	T **T** T

Chapter 4 Part G

1. Since we're given that the Boolean connectives are truth-functionally complete, all we need to do to show that the set $\{\neg, \wedge\}$ is also truth-functionally complete is to show that we can express $\vee$ using just $\neg$ and $\wedge$. A joint truth table will show that we can express $(P \vee Q)$ using $\neg(\neg P \wedge \neg Q)$:

P	Q	$(P \vee Q)$	$\neg\, (\, \neg\, P \wedge \neg\, Q)$
T	T	**T**	**T** F F F
T	F	**T**	**T** F F T
F	T	**T**	**T** T F F
F	F	**F**	**F** T T T

2. Similarly, we can show that the set $\{\neg, \vee\}$ is also truth-functionally complete by constructing a joint truth table showing that we can express $P \wedge Q$ using $\neg(\neg P \vee \neg Q)$:

P	Q	$(P \wedge Q)$	$\neg$	$(\neg$	P	$\vee$	$\neg$	$Q)$
T	T	**T**	**T**	F		F	F	
T	F	**F**	**F**	F		T	T	
F	T	**F**	**F**	T		T	F	
F	F	**F**	**F**	T		T	T	

Chapter 4 Part H

We can express the ternary connective ♣ using just the Boolean connectives with the following sentence schema: $\big((P \wedge Q) \vee (\neg P \wedge R)\big)$

13.5 Chapter 5 Solutions

Chapter 5 Part A

1. The mistake is at line 4: The →E rule allows us to derive the consequent of a conditional if we also have the antecedent. Line 4 attempts to derive the antecedent from the consequent.
2. The mistake is at line 5: the ∨I rule requires citing only one line, and here it's citing two. All we need is '∨I 3'.
3. The mistake is at line 2: The brackets are misplaced. The result of the ∧I rule does not change the placement of brackets.
4. There are two mistakes. First, there are missing brackets on line 4. Second, there is no rule that gets us the sentence $\neg\neg Mp$ from Mp in one step. As we'll see in chapter 6, this inference requires the use of subproofs.

Chapter 5 Part B

1.

1	$(Qee \wedge Ga)$	
2	$(Ha \wedge Rde)$	
3	Ga	∧E 1
4	Ha	∧E 2
5	$(Ga \wedge Ha)$	∧I 3, 4

2.

1	$(Hb \wedge Rde)$	
2	Hb	∧E 1
3	$(Hb \vee a = b)$	∨I 2

3.

1	$(Qee \to \neg Ga)$	
2	$(Hb \land Qee)$	
3	$\big(Ga \lor (Gb \land Gc)\big)$	
4	Qee	∧E 2
5	$\neg Ga$	→E 1, 4
6	$(Gb \land Gc)$	∨E 3, 5
7	Gc	∧E 6

4.

1	$(a \neq b \to a = c)$	
2	$\big(Pa \land (a = b \lor Rad)\big)$	
3	$a \neq b$	
4	Pa	∧E 2
5	$(a = b \lor Rad)$	∧E 2
6	Rad	∨E 3, 5
7	$a = c$	→E 1, 3
8	Pc	=E 4, 7
9	Rcd	=E 6, 7
10	$(Pc \land Rcd)$	∧I 8, 9
11	$b = b$	=I
12	$\big(b = b \land (Pc \land Rcd)\big)$	∧I 10, 11

Chapter 5 Part C

1.

1	Ba	
2	$\big(Ba \to (Cga \wedge De)\big)$	
3	$(Cga \wedge De)$	$\to$E 1, 2
4	Cga	$\wedge$E 3

2.

1	$\big((Tea \vee Mai) \to Uc\big)$	
2	Tea	
3	$(Tea \vee Mai)$	$\vee$I 2
4	Uc	$\to$E 1, 3

3.

1	Pa	
2	Qg	
3	$(Pa \wedge Qg)$	$\wedge$I 1, 2
4	$\big((Pa \wedge Qg) \vee Tjk\big)$	$\vee$I 3

4.

1	$\big((He \wedge Sea) \vee See\big)$	
2	$\big(Gb \to \neg(He \wedge Sea)\big)$	
3	$(Gb \wedge He)$	
4	Gb	$\wedge$E 3
5	$\neg(He \wedge Sea)$	$\to$E 2, 4
6	See	$\vee$E 1, 5
7	He	$\wedge$E 3
8	$(He \wedge See)$	$\wedge$I 6, 7

5.

1	$d \neq f$	
2	$(a = b \lor d = f)$	
3	$(a = b \to Tab)$	
4	$(Tbb \to \neg Hu)$	
5	$(Hu \lor Hi)$	
6	$a = b$	$\lor$E 1, 2
7	Tab	$\to$E 3, 6
8	Tbb	=E 6, 7
9	$\neg Hu$	$\to$E 4, 8
10	Hi	$\lor$E 5, 9
11	$c = c$	=I
12	$(Hi \land c = c)$	$\land$I 10, 11

6.

1	$\big((Te \lor Ga) \lor Lo\big)$	
2	$\big(Rbc \to \neg(Te \lor Ga)\big)$	
3	$\big(Lo \to (\neg Te \land \neg Ga)\big)$	
4	Raa	
5	$a = b$	
6	$b = c$	
7	Rba	=E 4, 5
8	Rbb	=E 5, 7
9	Rbc	=E 6, 8
10	$\neg(Te \lor Ga)$	$\to$E 2, 9
11	Lo	$\lor$E 1, 10
12	$(\neg Te \land \neg Ga)$	$\to$E 3, 11
13	$\neg Ga$	$\land$E 12

13.6 Chapter 6 Solutions

CHAPTER 6 PART A

1. The mistake is at line 4: We haven't yet discharged our assumption. When using the →I rule, we discharge the assumption.
2. The mistake is at line 4: The justification should use a dash, rather than a comma since we're citing the entire subproof. That is, it should read '→I 2-3'.
3. The mistake is at line 10: At this step, line 7 has been discharged and is no longer in force. So, it cannot be used in any justification outside of the subproof.
4. The mistake is at line 6: The proper use of →I requires that the subproof ends with the consequent of the conditional claim that it justifies.

CHAPTER 6 PART B

1.

1	$\big(Tu \to (Rc \land Oae)\big)$	
2	│ Tu	
3	│ $(Rc \land Oae)$	→E 1, 2
4	│ Rc	∧E 3
5	$(Tu \to Rc)$	→I 2–4

2.

1	$(Tu \to Rc)$	
2	│ Tu	
3	│ Rc	→E 1, 2
4	│ $(Rc \lor a = a)$	∨I 3
5	$\big(Tu \to (Rc \lor a = a)\big)$	→I 2–4

3.

1	Gb	
2	│ $a = b$	
3	│ $\big(Gb \land a = b\big)$	∧I 1, 2
4	$\big(a = b \to (Gb \land a = b)\big)$	→I 2–3

Chapter 6 Part C

1.

$$\begin{array}{r|ll}
1 & \neg Tu & \\ \hline
2 & \quad \vert\ \big(Tu \wedge a = b\big) & \text{for } \textit{reductio} \\
3 & \quad \vert\ Tu & \wedge\text{E } 2 \\
4 & \quad \vert\ \neg Tu & \text{R } 1 \\
5 & \neg\big(Tu \wedge a = b\big) & \neg\text{I } 2\text{–}4
\end{array}$$

2.

$$\begin{array}{r|ll}
1 & \neg(Ugg \vee Ta) & \\ \hline
2 & \quad \vert\ Ugg & \text{for } \textit{reductio} \\
3 & \quad \vert\ (Ugg \vee Ta) & \vee\text{I } 2 \\
4 & \quad \vert\ \neg(Ugg \vee Ta) & \text{R } 1 \\
5 & \neg Ugg & \neg\text{I } 2\text{–}4
\end{array}$$

3.

$$\begin{array}{r|ll}
1 & \neg Ta & \\
2 & \big(a = b \to Ta\big) & \\ \hline
3 & \quad \vert\ a = b & \text{for } \textit{reductio} \\
4 & \quad \vert\ Ta & \to\text{E } 2, 3 \\
5 & \quad \vert\ \neg Ta & \text{R } 1 \\
6 & a \neq b & \neg\text{I } 3\text{–}5
\end{array}$$

Chapter 6 Part D

The First Argument

Consider the following symbolization key:

Ix: x is Immoral
p: The Parisian cat-burnings
m: Modern pig framing
b: Buying pork

The inference is:

$$\begin{array}{rl}
 & Ip \\
 & (Ip \to Im) \\
 & (Im \to Ib) \\
\therefore & Ib
\end{array}$$

This is a valid inference, as the following natural deduction proof shows:

$$\begin{array}{r|ll}
1 & Ip & \\
2 & (Ip \to Im) & \\
3 & \underline{(Im \to Ib)} & \\
4 & Im & \to\text{E } 1, 2 \\
5 & Ib & \to\text{E } 3, 5
\end{array}$$

The Second Argument

Consider the following symbolization key:

Cxy: x was created by y
Tx: x is the same in all times and places
Sx: x is a social construction
g : God
m : Moral rules

The inference is:

$$\begin{array}{rl}
 & (Cmg \to Tm) \\
 & \neg Tm \\
 & (Cmg \lor Sm) \\
\therefore & Sm
\end{array}$$

This is a valid inference, as the following natural deduction proof shows:

$$\begin{array}{r|lll}
1 & (Cmg \to Tm) & & \\
2 & \neg Tm & & \\
3 & \underline{(Cmg \lor Sm)} & & \\
4 & \quad\vert\ \underline{\neg Sm} & & \\
5 & \quad\vert\ Cmg & & \lor\text{E } 4, 3 \\
6 & \quad\vert\ Tm & & \to\text{E } 5, 1 \\
7 & \quad\vert\ \neg Tm & & \text{R } 2 \\
8 & Sm & & \neg\text{E } 4\text{–}7
\end{array}$$

The Third Argument

Consider the following symbolization key:

Dx: x is on the desk in my office
Mx: x will miss their flight
Hx: x is at home
Fxy: x found y
p: My passport
i: I (me)
e: My wife

The inference is:

$$\begin{array}{ll}
 & (Dp \to Mi) \\
 & (\neg Hp \to Dp) \\
 & \big((Hp \to Fep) \wedge \neg Fep\big) \\
\therefore & Mi
\end{array}$$

This is a valid inference, as the following natural deduction proof shows:

$$\begin{array}{r|ll}
1 & (Dp \to Mi) & \\
2 & (\neg Hp \to Dp) & \\
3 & \big((Hp \to Fep) \wedge \neg Fep\big) & \\ \cline{2-2}
4 & (Hp \to Fep) & \wedge\text{E } 3 \\
5 & \quad \vert\ \underline{Hp} & \\
6 & \quad \vert\ Fep & \to\text{E } 5, 6 \\
7 & \quad \vert\ \neg Fep & \wedge\text{E } 3 \\
8 & \neg Hp & \neg\text{I } 5\text{–}7 \\
9 & Dp & \to\text{E } 2, 8 \\
10 & Mi & \to\text{E } 1, 9
\end{array}$$

Chapter 6 Part E

The first argument is valid, the second is not.

Valid: $Ra, \quad (Ra \to Tu) \quad \therefore \quad \big((Ra \lor Gc) \land Tu\big)$

1	Ra	
2	$(Ra \to Tu)$	
3	$(Ra \lor Gc)$	∨I 1
4	Tu	→E 1, 2
5	$\big((Ra \lor Gc) \land Tu\big)$	∧I 3, 4

Invalid: $Ra, \quad (Ra \to Tu) \quad \therefore \quad \big((Ra \land Gc) \land Tu\big)$

Ra	Gc	Tu	$(Ra \to Tu)$	$\big((Ra \land Gc) \land Tu\big)$
T	T	T	T	T **T**
T	T	F	F	T **F**
T	F	T	T	F **F** ★
T	F	F	F	F **F**
F	T	T	T	F **F**
F	T	F	T	F **F**
F	F	T	T	F **F**
F	F	F	T	F **F**

CHAPTER 6 PART F

1.

1	$\mid \underline{\big(Ra \to (b = c \land c = d)\big)}$	
2	$\mid\ \mid \underline{Ra}$	
3	$\mid\ \mid (b = c \land c = d)$	→E 1, 2
4	$\mid\ \mid b = c$	∧E 3
5	$\mid (Ra \to b = c)$	→I 2–4
6	$\mid\ \mid \underline{Ra}$	
7	$\mid\ \mid (b = c \land c = d)$	→E 1, 6
8	$\mid\ \mid c = d$	∧E 7
9	$\mid (Ra \to c = d)$	∧I 1–8
10	$\mid \big((Ra \to b = c) \land (Ra \to c = d)\big)$	∧I 5, 9

2.

1	$\mid \underline{\neg(Ra \lor Gb)}$	
2	$\mid\ \mid \underline{Ra}$	for *reductio*
3	$\mid\ \mid (Ra \lor Gb)$	∨I 2
4	$\mid\ \mid \neg(Ra \lor Gb)$	R 1
5	$\mid \neg Ra$	¬I 2–4
6	$\mid\ \mid \underline{Gb}$	for *reductio*
7	$\mid\ \mid (Ra \lor Gb)$	∨I 6
8	$\mid\ \mid \neg(Ra \lor Gb)$	R 1
9	$\mid \neg Gb$	¬I 6–8
10	$\mid (\neg Ra \land \neg Gb)$	∧I 5, 9

3.

1	$(\neg Ra \wedge \neg Gb)$		
2		$(Ra \vee Gb)$	for *reductio*
3		$\neg Ra$	$\wedge$E 1
4		Gb	$\vee$E 2, 3
5		$\neg Gb$	$\wedge$E 1
6	$\neg(Ra \vee Gb)$		$\neg$I 2–5

CHAPTER 6 PART G

1	$\mid \underline{\quad}$	
2	$\mid \; \mid \underline{\neg(Ra \wedge Gb)}$	
3	$\mid \; \mid \; \mid \underline{\neg(\neg Ra \vee \neg Gb)}$	for *reductio*
4	$\mid \; \mid \; \mid \; \mid \underline{Ra}$	for *reductio*
5	$\mid \; \mid \; \mid \; \mid \; \mid \underline{\neg Gb}$	for *reductio*
6	$\mid \; \mid \; \mid \; \mid \; \mid (\neg Ra \vee \neg Gb)$	∨I 5
7	$\mid \; \mid \; \mid \; \mid \; \mid \neg(\neg Ra \vee \neg Gb)$	R 3
8	$\mid \; \mid \; \mid \; \mid Gb$	¬E 5–7
9	$\mid \; \mid \; \mid \; \mid (Ra \wedge Gb)$	∧I 4, 8
10	$\mid \; \mid \; \mid \; \mid \neg(Ra \wedge Gb)$	R 2
11	$\mid \; \mid \; \mid \neg Ra$	¬I 4–10
12	$\mid \; \mid \; \mid (\neg Ra \vee \neg Gb)$	∨I 11
13	$\mid \; \mid \; \mid \neg(\neg Ra \vee \neg Gb)$	R 3
14	$\mid \; \mid (\neg Ra \vee \neg Gb)$	¬E 3–13
15	$\mid \; \mid \underline{(\neg Ra \vee \neg Gb)}$	
16	$\mid \; \mid \; \mid \underline{(Ra \wedge Gb)}$	for *reductio*
17	$\mid \; \mid \; \mid Ra$	∧E 16
18	$\mid \; \mid \; \mid \; \mid \underline{\neg Ra}$	for *reductio*
19	$\mid \; \mid \; \mid \; \mid Ra$	R 17
20	$\mid \; \mid \; \mid \neg\neg Ra$	¬I 18–19
21	$\mid \; \mid \; \mid \neg Gb$	∨E 15, 20
22	$\mid \; \mid \; \mid Gb$	∧E 16
23	$\mid \; \mid \neg(Ra \wedge Gb)$	¬I 16–22
24	$\mid (\neg(Ra \wedge Gb) \leftrightarrow (\neg Ra \vee \neg Gb))$	↔ 2–14, 15–23

13.7 Chapter 7 Solutions

CHAPTER 7 PART A

1. Ashni is a mathematician.
   Ma
2. Ashni is a philosopher.
   Pa
3. Ashni is either a mathematician or a philosopher.
   $(Ma \vee Pa)$
4. Ashni admires Ben.
   Dab
5. Ben admires Ashni.
   Dba
6. Ashni and Ben admire each other.
   $(Dab \wedge Dba)$
7. Ashni is a mathematician and she admires Ben.
   $(Ma \wedge Dab)$
8. Ashni and Ben are mathematicians who admire each other.
   $\big((Ma \wedge Mb) \wedge (Dab \wedge Dba)\big)$
9. Everyone is a mathematician. (i.e. Everyone at the party is a mathematician.)
   $\forall x Mx$
10. Everyone is either a mathematician or a philosopher.
    $\forall x(Mx \vee Px)$
11. Everyone admires Ashni.
    $\forall x Dxa$
12. Ashni admires everyone.
    $\forall x Dax$
13. Every mathematician admires Ashni.
    $\forall x(Mx \to Dxa)$
14. Everyone who admires Ashni is either a mathematician or a philosopher.
    $\forall x\big(Dxa \to (Mx \vee Px)\big)$

CHAPTER 7 PART B

Mx: x is a mouse
Ax: x is a mammal
Ex: x is an elephant
Bxy: x is bigger than y
j: Jerry
u: Jumbo

1. Jerry is a mouse.
   Mj

2. Jerry is a mammal.
   Aj
3. Jumbo is an elephant.
   Eu
4. Jumbo is bigger than Jerry.
   Buj
5. Every mouse is a mammal / Mice are mammals / A mouse is always a mammal.
   $\forall x(Mx \to Ax)$
6. Mice and elephants are mammals.
   $\big(\forall x(Mx \to Ax) \land \forall y(Ey \to Ay)\big)$

Chapter 7 Part C

1. Valid
   Everyone at the party is wearing red.
   Everyone who is wearing red is cool.
   ∴ Everyone at the party is cool.

2. Invalid
   Nobody at the party is wearing red.
   Nobody who is wearing red is cool.
   ∴ Nobody at the party is cool.

3. Invalid
   Everyone who likes Nickelback is cool.
   ∴ Everyone who's cool likes Nickelback.

4. Valid
   Everyone who likes Nickelback is cool.
   ∴ Anyone who isn't cool doesn't like Nickelback.

5. Valid
   Nobody who likes Nickelback is cool.
   ∴ Nobody who's cool likes Nickelback.

CHAPTER 7 PART D

English	**Symbols**
Someone loves Ashni.	$\exists x\, Lxa$
Ben and someone love each other.	$\boldsymbol{\exists x\,(Lbx \wedge Lxb)}$
Someone who is dancing and chatting likes Ben.	$\exists x\big((Dx \wedge Cx) \wedge Lxb\big)$
Someone's dancing and someone's not.	$(\exists x\, Dx \wedge \exists y\, \neg Dy)$
If someone is chatting, then someone is dancing.	$\boldsymbol{(\exists x\, Cx \rightarrow \exists y\, Dy)}$
Either everyone is dancing, or someone is not dancing.	$\boldsymbol{(\forall x\, Dx \vee \exists y\, \neg Dy)}$

CHAPTER 7 PART E

1. Amos, Bouncer, and Cleo all live at the zoo.
   $\big(Zs \wedge (Zb \wedge Zc)\big)$
2. Bouncer is a reptile, but not an alligator.
   $(Rb \wedge \neg Ab)$
3. If Cleo loves Bouncer, then Bouncer is a monkey.
   $(Lcb \rightarrow Mb)$
4. If both Bouncer and Cleo are alligators, then Amos loves them both.
   $\big((Ab \wedge Ac) \rightarrow (Lsb \wedge Lsc)\big)$
5. Some reptile lives at the zoo.
   $\exists x\,(Rx \wedge Zx)$
6. Every alligator is a reptile.
   $\forall x\,(Ax \rightarrow Rx)$
7. Any animal that lives at the zoo is either a monkey or an alligator.
   $\forall x\big(Zx \rightarrow (Mx \vee Ax)\big)$
8. There are reptiles which are not alligators.
   $\exists x\,(Rx \wedge \neg Ax)$
9. Cleo loves a reptile.
   $\exists x\,(Rx \wedge Lcx)$

10. Bouncer loves all the monkeys that live at the zoo.
    $\forall x\big((Mx \wedge Zx) \rightarrow Lbx\big)$
11. All the monkeys that Amos loves love him back.
    $\forall x\big((Mx \wedge Lsx) \rightarrow Lxs\big)$
12. If any animal is a reptile, then Amos is.
    $(\exists x\, Rx \rightarrow Rs)$
13. If any animal is an alligator, then it is a reptile.
    $\forall x\,(Ax \rightarrow Rx)$
14. Every monkey that Cleo loves is also loved by Amos.
    $\forall x\big((Mx \wedge Lcx) \rightarrow Lsx\big)$
15. There is a monkey that loves Bouncer, but sadly Bouncer does not reciprocate this love.
    $\exists x\big((Mx \wedge Lxb) \wedge \neg Lbx\big)$

Chapter 7 Part F

Baralipton: All *B*s are *C*s. All *A*s are *B*s. ∴ Some *C* is *A*.
$\forall x\,(Bx \rightarrow Cx),\ \ \forall y\,(Ay \rightarrow By)\ \therefore\ \exists z\,(Cz \wedge Az)$

Barbara: All *B*s are *C*s. All *A*s are *B*s. ∴ All *A*s are *C*s.
$\forall x\,(Bx \rightarrow Cx),\ \ \forall y\,(Ay \rightarrow By)\ \therefore\ \forall z\,(Az \rightarrow Cz)$

Baroco: All *C*s are *B*s. Some *A* is not *B*. ∴ Some *A* is not *C*.
$\forall x\,(Cx \rightarrow Bx),\ \ \exists y\,(Ay \wedge \neg By)\ \therefore\ \exists z\,(Az \wedge \neg Cz)$

Bocardo: Some *B* is not *C*. All *A*s are *B*s. ∴ Some *A* is not *C*.
$\exists x\,(Bx \wedge \neg Cx),\ \ \forall y\,(Ay \rightarrow By)\ \therefore\ \exists z\,(Az \wedge \neg Cz)$

Celantes: No *B*s are *C*s. All *A*s are *B*s. ∴ No *C*s are *A*s.
$\forall x\,(Bx \rightarrow \neg Cx),\ \ \forall y\,(Ay \rightarrow By)\ \therefore\ \forall z\,(Cz \rightarrow \neg Az)$

Calarent: No *B*s are *C*s. All *A*s are *B*s. ∴ No *A*s are *C*s.
$\forall x\,(Bx \rightarrow \neg Cx),\ \ \forall y\,(Ay \rightarrow By)\ \therefore\ \forall z\,(Az \rightarrow \neg Cz)$

Cemestres: No *C*s are *B*s. No *A*s are *B*s. ∴ No *A*s are *C*s.
$\forall x\,(Cx \rightarrow \neg Bx),\ \ \forall y\,(Ay \rightarrow \neg By)\ \ \therefore \forall z\,(Az \rightarrow \neg Cz)$

Cesare: No *C*s are *B*s. All *A*s are *B*s. ∴ No *A*s are *C*s.
$\forall x\,(Cx \rightarrow \neg Bx),\ \ \forall y\,(Ay \rightarrow By)\ \therefore\ \forall z\,(Az \rightarrow \neg Cz)$

Dabitis: All *B*s are *C*s. Some *A* is *B*. ∴ Some *C* is *A*.
$\forall x\,(Bx \rightarrow Cx),\ \ \exists y\,(Ay \wedge By)\ \therefore\ \exists z\,(Cz \wedge Az)$

Darii: All *B*s are *C*s. Some *A* is *B*. ∴ Some *A* is *C*.
$\forall x\,(Bx \rightarrow Cx),\ \ \exists y\,(Ay \wedge By)\ \therefore\ \exists z\,(Az \wedge Cz)$

Datisi: All *B*s are *C*s. Some *B* is *A*. ∴ Some *C* is *A*.
$\forall x\,(Bx \rightarrow Cx),\quad \exists y\,(By \wedge Ay)\quad \therefore\quad \exists z\,(Cz \wedge Az)$

Disamis: Some *A* is *B*. All *A*s are *C*s. ∴ Some *B* is *C*.
$\exists x\,(Ax \wedge Bx),\quad \forall y\,(Ay \rightarrow Cy)\quad \therefore\quad \exists z\,(Bz \wedge Cz)$

Ferison: No *B*s are *C*s. Some *B* is *A*. ∴ Some *A* is not *C*.
$\forall x\,(Bx \rightarrow \neg Cx),\quad \exists y\,(By \wedge Ay)\quad \therefore\quad \exists z\,(Az \wedge \neg Cz)$

Ferio: No *B*s are *C*s. Some *A* is *B*. ∴ Some *A* is not *C*.
$\forall x\,(Bx \rightarrow \neg Cx),\quad \exists y\,(Ay \wedge By)\quad \therefore\quad \exists z\,(Az \wedge \neg Cz)$

Festino: No *C*s are *B*s. Some *A* is *B*. ∴ Some *A* is not *C*.
$\forall x\,(Cx \rightarrow \neg Bx),\quad \exists y\,(Ay \wedge By)\quad \therefore\quad \exists z\,(Az \wedge \neg Cz)$

Frisesomorum: Some *B* is *C*. No *A*s are *B*s. ∴ Some *C* is not *A*.
$\exists x\,(Bx \wedge Cx),\quad \forall y\,(Ay \rightarrow \neg By)\quad \therefore\quad \exists z\,(Cz \wedge \neg Az)$

CHAPTER 7 PART G

1. Bertie is a dog who likes samurai movies.
   $(Db \wedge Sb)$
2. Bertie, Emerson, and Fergis are all dogs.
   $\big((Db \wedge De) \wedge Df\big)$
3. Emerson is larger than Bertie, and Fergis is larger than Emerson.
   $(Leb \wedge Lfe)$
4. All dogs like samurai movies.
   $\forall x\,(Dx \rightarrow Sx)$
5. Only dogs like samurai movies.
   $\forall x\,(Sx \rightarrow Dx)$
6. There is a dog that is larger than Emerson.
   $\exists x\,(Dx \wedge Lxe)$
7. If there is a dog larger than Fergis, then there is a dog larger than Emerson.
   $\big(\exists x\,(Dx \wedge Lxf) \rightarrow \exists y\,(Dy \wedge Lye)\big)$
8. No animal that likes samurai movies is larger than Emerson.
   $\forall x\,(Sx \rightarrow \neg Lxe)$
9. No dog is larger than Fergis.
   $\forall x\,(Dx \rightarrow \neg Lxf)$
10. Any animal that dislikes samurai movies is larger than Bertie.
    $\forall x\,(\neg Sx \rightarrow Lxb)$
11. There is an animal that is between Bertie and Emerson in size.
    $\exists x\,(Lxb \wedge Lex)$
12. There is no dog that is between Bertie and Emerson in size.
    $\forall x\,\big(Dx \rightarrow \neg(Lxb \wedge Lex)\big)$
13. No dog is larger than itself.
    $\forall x\,(Dx \rightarrow \neg Lxx)$

CHAPTER 7 PART H

1. Consider the following symbolization key:

   UD: Things on my desk
   Cx: x is a computer
   Ex: x escapes my attention

   Here is a translation in FOL:

   $\neg\exists x\, Ex, \quad \exists x\, Cx \quad \therefore \quad \exists x\,(Cx \wedge \neg Ex)$

2. Consider the following symbolization key:

   UD: Everything
   Bx: x is black and white
   Dx: x is my dream
   Sx: x is an old TV show

   Here is a translation in FOL:

   $\forall x\,(Dx \rightarrow Bx), \quad \forall x\,(Sx \rightarrow Bx) \quad \therefore \quad \exists x\,(Dx \wedge Sx)$

3. Consider the following symbolization key:

   UD: People
   Ax: x has been to Australia
   Zx: x has been to a zoo
   Kx: x has seen a kangaroo
   h: Holmes
   t: Watson

   Here is a translation in FOL:

   $\neg(Ah \vee At), \quad \forall x\Big(Kx \rightarrow (Ax \vee Zx)\Big), \quad (\neg Kt \wedge Kh) \quad \therefore \quad Zh$

4. Consider the following symbolization key:

   UD: People
   Ex: x expects the Spanish Inquisition
   Kx: x knows the troubles I've seen

   Here is a translation in FOL:

   $\forall x\,\neg Ex, \quad \forall x\,\neg Kx, \quad \therefore \quad \forall x\,(Ex \rightarrow Kx)$

5. Consider the following symbolization key:

   UD: People
   Bx: x is a baby
   Cx: x can manage a crocodile
   Ix: x is illogical
   e: Berthold

   Here is a translation in FOL:

   $\forall x\,(Bx \rightarrow Ix), \quad \forall x\,(Ix \rightarrow \neg Cx), \quad Be \quad \therefore \quad \neg Ce$

CHAPTER 7 PART I

1. Boris has never tried any candy.
   $\forall x\, \neg Tbx$
2. Marzipan is always made with sugar.
   $\forall x\, (Mx \rightarrow Sx)$
3. Some candy is sugar-free.
   $\exists x\, \neg Sx$
4. No candy is better than itself.
   $\neg \exists x\, Bxx$
5. Boris has never tried sugar-free chocolate.
   $\forall x \big((Cx \land \neg Sx) \rightarrow \neg Tbx\big)$
6. Boris has tried marzipan and chocolate, but never together.
   $\big([\exists x\, (Mx \land Tbx) \land \exists y\, (Cy \land Tby)] \land \neg \exists z\, [(Mz \land Cz) \land Tbz]\big)$

CHAPTER 7 PART J

1. All the food is on the table.
   $\forall x\, (Fx \rightarrow Tx)$
2. If the guacamole has not run out, then it is on the table.
   $(\neg Rg \rightarrow Tg)$
3. Everyone likes the guacamole.
   $\forall x\, (Px \rightarrow Lxg)$
4. If anyone likes the guacamole, then Eli does.
   $(\exists x\, Lxg \rightarrow Leg)$
5. Francesca only likes the dishes that have run out.
   $\forall x \big((Fx \land Lfx) \rightarrow \neg Rx\big)$
6. Francesca likes no one, and no one likes Francesca.
   $(\forall x\, \neg Lfx \land \forall y\, \neg Lyf)$
7. Eli likes anyone who likes the guacamole.
   $\forall x\, (Lxg \rightarrow Lex)$

CHAPTER 7 PART K

1. All of Patrick's children are ballet dancers.
   $\forall x\, (Cxp \rightarrow Dx)$
2. Jane is Patrick's daughter.
   $(Fj \land Cjp)$
3. Patrick has a daughter.
   $\exists x\, (Cxp \land Fx)$
4. Jane is an only child.
   $\neg \exists x\, Sxj$
5. All of Patrick's daughters dance ballet.
   $\forall x \big((Cxp \land Fx) \rightarrow Dx\big)$

6. Patrick has no sons.
   $\neg\exists x\,(Mx \wedge Cxp)$

Chapter 7 Part L

Bound: $\boxed{}$
Free: _

1. $(\exists x\, L \boxed{\text{x}}\, \underline{y} \wedge \forall y\, L \boxed{\text{y}}\, \underline{x})$
   Bound variables: the 'x' of the first instantiation of *Lxy* and the 'y' of the second instantiation of *Lxy*.
   Free variables: the 'y' of the first instantiation of *Lxy* and the 'x' of the second instantiation of *Lxy*.
2. $(\forall x\, A \boxed{\text{x}} \wedge B\, \underline{x})$
   Bound variables: the 'x' of *Ax*.
   Free variables: the 'x' of *Bx*.
3. $\big(\forall x\, (A \boxed{\text{x}} \wedge B \boxed{\text{x}}) \wedge \forall y\, (A\underline{x} \wedge D \boxed{\text{y}})\big)$
   Bound variables: the 'x' of the first instantiation of *Ax*, the 'x' of Bx, and the 'y' of *Dy*.
   Free variables: the 'x' of the second instantiation of *Ax*.
4. $\big(\forall x\, \exists y\, [R \boxed{\text{xy}} \rightarrow (J\underline{z} \wedge K \boxed{\text{x}})] \vee R\underline{yx}\big)$
   Bound variables: the 'x' and 'y' of the first instantiation of *Rxy* and the 'x' of *Kx*.
   Free variables: the 'z' of *Jz* and the 'x' and 'y' of *Rxy*.
5. $\big(\forall x\, (M\underline{y} \leftrightarrow L\underline{y} \boxed{\text{x}}) \wedge \exists y\, L\underline{z} \boxed{\text{y}}\big)$
   Bound variables: the *x* of *Lyx* and the *y* of *Lzy*.
   Free variables: the *y* of *My*, the *y* of *Lyx*, and the *z* of *Lzy*.

13.8 Chapter 8 Solutions

CHAPTER 8 PART A

1. Every step contains a mistake. When using the E rule we need to substitute every instance of a variable with a constant. ‘x’ is not a constant; it is a variable.
2. The mistake is at step 4. The first premise is a material conditional claim, not a universal generalization. So, we cannot use ∀E on step 1.
3. The mistake is on step 5. The ∀I rule requires that the constant *c* on which we are generalizing doesn’t appear in any undischarged assumption. But *a* appears in the premises, which count as undischarged assumptions.

CHAPTER 8 PART B

1.

$$\begin{array}{r|ll}
1 & \forall x(Cx \to Dx) & \\
2 & Ca & \\
\cline{2-2}
3 & (Ca \to Da) & \forall\text{E } 1 \\
4 & Da & \to\!\text{E } 2, 3
\end{array}$$

2.

$$\begin{array}{r|ll}
1 & \forall x\big((Cx \wedge Lxa) \to Dx\big) & \\
2 & Cb & \\
3 & Lba & \\
\cline{2-2}
4 & \big((Cb \wedge Lba) \to Db\big) & \forall\text{E } 1 \\
5 & (Cb \wedge Lba) & \wedge\text{I } 2, 3 \\
6 & Db & \to\!\text{E } 4, 5
\end{array}$$

3.

1	$\forall xCx$	
2	$\forall xDx$	
3	Ca	$\forall$E 1
4	Da	$\forall$E 2
5	$(Ca \wedge Da)$	$\wedge$I 3, 4

Chapter 8 Part C

1.

1	$\forall x(Cx \wedge Dx)$	
2	$(Ce \wedge De)$	$\forall$E 1
3	Ce	$\wedge$E 2
4	$\forall xCx$	$\forall$I 3

2.

1	$\forall x(Cx \rightarrow Dx)$	
2	$\forall xCx$	
3	$(Ce \rightarrow De)$	$\forall$E 1
4	Ce	$\forall$E 2
5	De	$\rightarrow$E 3, 4
6	$\forall xDx$	$\forall$I 5

3.

1	$\forall x(Cx \to Dx)$	
2	$(Ce \to De)$	$\forall$E 1
3	$\quad \neg De$	for conditional proof
4	$\quad\quad Ce$	for *reductio*
5	$\quad\quad De$	$\to$E 2, 4
6	$\quad\quad \neg De$	R 3
7	$\quad \neg Ce$	$\neg$I 4–6
8	$(\neg De \to \neg Ce)$	$\to$I 3–7
9	$\forall x(\neg Dx \to \neg Cx)$	$\forall$I 8

CHAPTER 8 PART D

1. Invalid

2. Valid:

1	$\forall x\,(Px \wedge Qx)$	
2	$(Pa \wedge Qa)$	$\forall$E 1
3	Pa	$\wedge$E 2
4	Qa	$\wedge$E 2
5	$\forall x\,Px$	$\forall$I 3
6	$\forall x\,Qx$	$\forall$I 4
7	$(\forall x\,Px \wedge \forall x\,Qx)$	$\wedge$I 5, 6

3. Valid:

1	$\mid (\forall x\, Px \lor \forall x\, Qx)$	
2	$\mid\; \mid \neg(Pa \lor Qa)$	for *reductio*
3	$\mid\; \mid\; \mid \forall x\, Px$	for *reductio*
4	$\mid\; \mid\; \mid Pa$	∀E 3
5	$\mid\; \mid\; \mid (Pa \lor Qa)$	∨I 4
6	$\mid\; \mid\; \mid \neg(Pa \lor Qa)$	R 2
7	$\mid\; \mid \neg\forall x\, Px$	¬I 3–6
8	$\mid\; \mid \forall x\, Qx$	∨E 7, 1
9	$\mid\; \mid Qa$	∀E 8
10	$\mid\; \mid (Pa \lor Qa)$	∨I 9
11	$\mid\; \mid \neg(Pa \lor Qa)$	R 2
12	$\mid (Pa \lor Qa)$	¬E 2–11
13	$\mid \forall x\, (Px \lor Qx)$	∀I 12

4. Valid:

1	$\mid (\forall x\, Px \land \forall x\, Qx)$	
2	$\mid \forall x\, Px$	∧E 1
3	$\mid \forall x\, Qx$	∧E 1
4	$\mid Pa$	∀E 2
5	$\mid Qa$	∀E 3
6	$\mid (Pa \land Qa)$	∧I 4, 5
7	$\mid \forall x\, (Px \land Qx)$	∀I 6

13.9 Chapter 9 Solutions

Chapter 9 Part A

1. The mistake is at step 2: the sentence $\exists y\, Py$ is not a reiteration of $\exists x\, Px$.
2. The mistake is at step 6: the constant k that was used as a proxy cannot appear outside of the subproof in which it was introduced.
3. The mistake is at step 3: since the constant a is already in force, it cannot be used as a proxy.
4. The mistake is at step 7: the $\exists$E rule requires that you cite the step containing the existential generalization as well as the subproof.

Chapter 9 Part B

1.

1	Da	
2	Ca	
3	$(Ca \wedge Da)$	$\wedge$I 1, 2
4	$\exists x(Cx \wedge Dx)$	$\exists$I 3

2.

1	Da	
2	$(Da \vee Ca)$	$\vee$I 1
3	$\exists x(Dx \vee Cx)$	$\exists$I 2

3.

1	$\exists x(Cx \wedge Dx)$		
2		$(Cm \wedge Dm)$	for existential instantiation
3		Cm	$\wedge$E 2
4		$\exists x Cx$	$\exists$I 3
5	$\exists x Cx$		$\exists$E 1, 2–4

4.

1	$\exists x \neg Cx$	
2	$\forall x(Dx \vee Cx)$	
3	$\quad \neg Cm$	for existential instantiation
4	$\quad (Dm \vee Cm)$	$\forall$E 2
5	$\quad Dm$	$\vee$E 3, 4
6	$\quad \exists x Dx$	$\exists$I 5
7	$\exists x Dx$	$\exists$I 1, 3–6

Chapter 9 Part C

1.

1	$\neg Da$	
2	$(\exists x Cx \rightarrow Da)$	
3	$\quad Cb$	for *reductio*
4	$\quad \exists x Cx$	$\exists$I 3
5	$\quad Da$	$\rightarrow$E 2, 4
6	$\quad \neg Da$	R 1
7	$\neg Cb$	$\neg$I 3–6
8	$\forall x \neg Cx$	$\forall$I 7

2. This argument is invalid. Let Dx mean 'x is a dog' and let Cx mean 'x is a cat'. Now let 'a' name my pet dog Alvin. The premises state that no dog is a cat, and that Alvin is not a cat. Both of these are true, but the conclusion is false: Alvin is a dog.

CHAPTER 9 PART D

1.

1	$\neg\forall x\, Px$	
2	$\quad\neg\exists x\,\neg Px$	for *reductio*
3	$\quad\quad\neg Pa$	for *reductio*
4	$\quad\quad\exists x\,\neg Px$	∃I 3
5	$\quad\quad\neg\exists x\,\neg Px$	R 2
6	$\quad Pa$	¬E 3–5
7	$\quad\forall x\, Px$	∀I 6
8	$\quad\neg\forall x\, Px$	R 1
9	$\exists x\,\neg Px$	¬E 2–8

2.

1	$\neg\exists x\, Px$	
2	$\quad Pa$	for *reductio*
3	$\quad\exists x\, Px$	∃I 2
4	$\quad\neg\exists x\, Px$	R 1
5	$\neg Pa$	¬I 2–4
6	$\forall x\,\neg Px$	∀I 5

CHAPTER 9 PART E

1.

1	$\mid \exists x\,(Px \lor Qx)$	
2	$\mid \mid (Pa \lor Qa)$	for existential instantiation
3	$\mid \mid \mid \neg(\exists x\,Px \lor \exists x\,Qx)$	for *reductio*
4	$\mid \mid \mid \mid \exists x\,Px$	for *reductio*
5	$\mid \mid \mid \mid (\exists x\,Px \lor \exists x\,Qx)$	∨I 4
6	$\mid \mid \mid \mid \neg(\exists x\,Px \lor \exists x\,Qx)$	R 3
7	$\mid \mid \mid \neg\exists x\,Px$	¬I 4–6
8	$\mid \mid \mid \mid Pa$	for *reductio*
9	$\mid \mid \mid \mid \exists x\,Px$	∃I 8
10	$\mid \mid \mid \mid \neg\exists x\,Px$	R 7
11	$\mid \mid \mid \neg Pa$	¬I 8–10
12	$\mid \mid \mid Qa$	∨E 2, 11
13	$\mid \mid \mid \mid \exists x\,Qx$	for *reductio*
14	$\mid \mid \mid \mid (\exists x\,Px \lor \exists x\,Qx)$	∨I 13
15	$\mid \mid \mid \mid \neg(\exists x\,Px \lor \exists x\,Qx)$	R 3
16	$\mid \mid \mid \neg\exists x\,Qx$	¬I 13–15
17	$\mid \mid \mid \exists x\,Qx$	∃I 12
18	$\mid \mid (\exists x\,Px \lor \exists x\,Qx)$	¬E 3–17
19	$\mid (\exists x\,Px \lor \exists x\,Qx)$	∃E 1, 2–18

2.

1	$(\exists x\, Px \lor \exists x\, Qx)$	
2	$\neg\exists x\, (Px \lor Qx)$	for *reductio*
3	$\exists x\, Px$	for *reductio*
4	Pa	for existential instantiation
5	$(Pa \lor Qa)$	∨I 4
6	$\exists x\, (Px \lor Qx)$	∃I 5
7	$\exists x\, (Px \lor Qx)$	∃E 3, 4–6
8	$\neg\exists x\, (Px \lor Qx)$	R 2
9	$\neg\exists x\, Px$	¬I 3–8
10	$\exists x\, Qx$	∨E 1, 9
11	Qb	for existential instantiation
12	$(Pb \lor Qb)$	∨I 11
13	$\exists x\, (Px \lor Qx)$	∃I 12
14	$\exists x\, (Px \lor Qx)$	∃E 10, 11–13
15	$\neg\exists x\, (Px \lor Qx)$	R 2
16	$\exists x\, (Px \lor Qx)$	¬E 2–15

3.

1	$\exists x\, (Px \land Qx)$	
2	$(Pa \land Qa)$	for existential instantiation
3	Pa	∧E 2
4	Qa	∧E 2
5	$\exists x\, Px$	∃I 3
6	$\exists x\, Qx$	∃I 4
7	$(\exists x\, Px \land \exists x\, Qx)$	∧I 5, 6
8	$(\exists x\, Px \land \exists x\, Qx)$	∃E 1, 2–7

4. This inference is invalid. Consider the following symbolization key:

UD: Things at a brunch
Px: x is a philosopher
Qx: x is a quince

Here is an FO counterexample. Suppose you've invited Phillipa Foot over for brunch, and in the fruit bowl are some delicious and tart quinces. In this circumstance, the premise is true: there is a philosopher, and there is quince. But the conclusion is false: there is no philosopher that's a quince.

CHAPTER 9 PART F

To show that →I is sound in t:

Suppose there is a proof p in t with assumptions $A_1, ..., A_k$ that allows you to derive at step k the sentence $(P \rightarrow Q)$ from an application of →I to an earlier subproof with assumption P and that ends with Q, but where $(P \rightarrow Q)$ is not in fact a tautological consequence of the assumptions in force at k. This means that there is a row in a joint truth table that assigns a 'T' to each of $A_1, ..., A_k$ and an 'F' to $(P \rightarrow Q)$. Call this row $\hbar$. Since $(P \rightarrow Q)$ receives an 'F' in $\hbar$, this means that P receives a 'T' and Q receives an 'F' in $\hbar$. Let step k be the first invalid step in p. Since k is the first invalid step, the step $k-1$ that derives Q from $A_1, ..., A_k$ and the further assumption P is tautologically valid. This means that there is no row that assigns a 'T' to each of $A_1, ..., A_k$ and P, and an 'F' to Q. But now we've reached a contradiction, since we've determined that row $\hbar$ assigns a 'T' to each of $A_1, ..., A_k$ and P, and an 'F' to Q. our initial assumption must be false: →I cannot be responsible for the first invalid step in p. →I is therefore sound in t.

13.10 Chapter 10 Solutions

Chapter 10 Part A

1. All of the black pieces are in front of all the white pieces.
   $\forall x\, \forall y \big((Bx \land Wx) \to Fxy\big)$
2. Some rook is to the left of a knight.
   $\exists x\, \exists y \big((Rx \land Ky) \land Lxy\big)$
3. All white pawns are in the same column.
   $\forall x\, \forall y \Big[\big((Wx \land Px) \land (Wy \land Py)\big) \to Cxy\Big]$
4. Not all black pawns are in the same column.
   $\neg \forall x\, \forall y \Big[\big((Bx \land Px) \land (By \land Py)\big) \to Cxy\Big]$
5. Every rook is in a different row than every other rook.
   $\forall x\, \forall y \Big[\big((Rx \land Ry) \land x \neq y\big) \to \neg Oxy\Big]$
6. Every pawn is in a different column than every other pawn.
   $\forall x\, \forall y \Big[\big((Px \land Py) \land x \neq y\big) \to \neg Cxy\Big]$
7. Different knights are in the same column.
   $\exists x\, \exists y \Big[\big((Kx \land Ky) \land x \neq y\big) \land Cxy\Big]$
8. There are no different rooks in the same row.
   $\forall x\, \forall y \Big[\big((Rx \land Ry) \land Oxy\big) \to x = y\Big]$

Chapter 10 Part B

1. Every pawn is to the left of some rook.
2. There is a pawn with no white rook in its column.
3. No black knight has a white pawn in front of it.
4. Some rook and some pawn aren't in the same column.
5. There is at least two rooks.
6. There is at least two pieces on the board.
7. If there is some piece in the same column as some piece, then every piece is in the same row.
8. Every rook in front of every pawn is in the same column as some knight.

Chapter 10 Part C

1. Every green frog is smaller than a brown dog.
   $\forall x \Big[(Gx \land Fx) \to \exists y \big((By \land Dy) \land Sxy\big)\Big]$
2. Some frog is bigger than every mouse.
   $\exists x \big(Fx \land \forall y\, (My \to Ixy)\big)$
3. Nothing is bigger than everything.
   $\neg \exists x\, \forall y\, Ixy$

4. Every frog bigger than every mouse is green.
   $\forall x\,\forall y\,\big[\big((Fx \land My) \land Ixy\big) \rightarrow Gx\big]$
5. Nothing smaller than a frog is bigger than a dog.
   $\neg\exists x\,\exists y\,\exists z\,\big((Fy \land Dz) \land (Sxy \land Ixz)\big)$
6. Some dog is smaller than some mouse.
   $\exists x\,\exists y\,\big((Dx \land My) \land Sxy\big)$
7. No mouse is bigger than every dog.
   $\forall x\,\forall y\,\big((Mx \land Dy) \rightarrow \neg Ixy\big)$
8. Mighty Mouse is bigger than any other mouse.
   $\forall x\,\big((Mx \land x \neq m) \rightarrow Imx\big)$

CHAPTER 10 PART D

1.

1	$\exists y\,\forall x\,Lyx$	
2	$\quad \forall x\,Lax$	for existential instantiation
3	$\quad Lab$	$\forall$E 2
4	$\quad \exists y\,Lyb$	$\exists$I 3
5	$\exists y\,Lyb$	$\exists$E 1, 2–4
6	$\forall x\,\exists y\,Lyx$	$\forall$I 5

2.

1	$\exists x\,(Dx \land \exists y\,Rxy)$	
2	$\forall x\,(\exists y\,Rxy \rightarrow Cx)$	
3	$\quad (Da \land \exists y\,Ray)$	for existential instantiation
4	$\quad Da$	$\land$E 3
5	$\quad (\exists y\,Ray \rightarrow Ca)$	$\forall$E 2
6	$\quad \exists y\,Ray$	$\land$E 3
7	$\quad Ca$	$\rightarrow$E 5, 6
8	$\quad \exists x\,Cx$	$\exists$I 7
9	$\exists x\,Cx$	$\exists$E 1, 2–8

3.

1	$\forall x \big(Px \rightarrow \exists y\,(Ry \land Lxy)\big)$	
2	$\exists x\, Px$	
3	$\quad Pa$	for existential instantiation
4	$\quad \big(Pa \rightarrow \exists y\,(Ry \land Lay)\big)$	∀E 1
5	$\quad \exists y\,(Ry \land Lay)$	→E 3, 4
6	$\quad\quad (Rb \land Lab)$	for existential instantiation
7	$\quad\quad Rb$	∧E 6
8	$\quad\quad \exists x\, Rx$	∃I 7
9	$\quad \exists x\, Rx$	∃E 5, 6–8
10	$\exists x\, Rx$	∃E 2, 3–9

4.

1	$\mid \forall x\big(Px \to \forall y\,(Qy \to Ryx)\big)$	
2	$\mid \underline{\forall x\,\forall y\,(Rxy \to Txy)}$	
3	$\mid \mid \underline{Pa}$	for conditional proof
4	$\mid \mid \mid \underline{Qb}$	for conditional proof
5	$\mid \mid \mid \big(Pa \to \forall y\,(Qy \to Rya)\big)$	∀E 1
6	$\mid \mid \mid \forall y\,(Qy \to Rya)$	→E 3, 5
7	$\mid \mid \mid (Qb \to Rba)$	∀E 6
8	$\mid \mid \mid \forall y\,(Rby \to Tby)$	∀E 2
9	$\mid \mid \mid (Rba \to Tba)$	∀E 8
10	$\mid \mid \mid Rba$	→E 4, 7
11	$\mid \mid \mid Tba$	→E 9, 10
12	$\mid \mid (Qb \to Tba)$	→I 4–11
13	$\mid \mid \forall y\,(Qy \to Tya)$	∀I 12
14	$\mid \big(Pa \to \forall y\,(Qy \to Tya)\big)$	→I 3–13
15	$\mid \forall x\big(Px \to \forall y\,(Qy \to Tyx)\big)$	∀I 14

5.

1	$\mid \underline{\exists x\big(Dx \wedge \neg\forall y\,(Cy \to Bxy)\big)}$	
2	$\mid \mid \underline{\big(Da \wedge \neg\forall y\,(Cy \to Bay)\big)}$	for existential instantiation
3	$\mid \mid Da$	∧E 2
4	$\mid \mid \neg\forall y\,(Cy \to Bay)$	∧E 2
5	$\mid \mid \mid \underline{\forall x\big(Dx \to \forall y\,(Cy \to Bxy)\big)}$	for *reductio*
6	$\mid \mid \mid \big(Da \to \forall y\,(Cy \to Bay)\big)$	∀E 5
7	$\mid \mid \mid \forall y\,(Cy \to Bay)$	→E 3, 6
8	$\mid \mid \mid \neg\forall y\,(Cy \to Bay)$	R 4
9	$\mid \mid \neg\forall x\big(Dx \to \forall y\,(Cy \to Bxy)\big)$	¬I 5–8
10	$\mid \neg\forall x\big(Dx \to \forall y\,(Cy \to Bxy)\big)$	∃E 1, 2–9

13.11 Chapter 11 Solutions

Chapter 11 Part A

Consider the following symbolization key:

UD: Animals at the zoo
Lx: x is a lion.
Rx: x is a rhino.
Sx: x is sleeping.
o: Rodney

The sentences can be translated thusly:

1. There's a rhino at the zoo.
   $\exists x\, Rx$
2. Rodney is the only rhino at the zoo.
   $\big(Ro \land \forall x\,(Rx \to x = o)\big)$
3. There is only one rhino at the zoo.
   $\exists\big(Rx \land \forall y\,(Ry \to y = x)\big)$
4. There are at least two lions at the zoo.
   $\exists x\,\exists y\,\big((Lx \land Ly) \land x \neq y\big)$
5. There are at most two lions at the zoo.
   $\forall x\,\forall y\,\forall z\,\Big[\big((Lx \land Ly) \land Lz\big) \to \big((x = y \lor y = z) \lor x = z\big)\Big]$
6. There are exactly two lions at the zoo.
   $\exists x\,\exists y\,\Big[\big((Lx \land Ly\big) \lor x \neq y) \land \forall z\big(Lz \to (z = x \lor z = y)\big)\Big]$
7. A lion is sleeping.
   $\exists x\,(Lx \land Sx)$
8. The lion is sleeping.
   $\exists x\,\Big[Lx \land \big(\forall y\,(Ly \to y = x) \land Sx\big)\Big]$

Chapter 11 Part B

1. Ashni is the only partygoer who loves Ben.
2. The only partygoer who loves Ben is Ben himself.
3. At least two partygoers are dancing.
4. The partygoer who loves Ashni isn't dancing.
5. There are exactly two people at the party.

CHAPTER 11 PART C

1.

1	$\exists x\, \exists y \big((Px \land Py) \land x \neq y\big)$	
2	$\mid\ \exists y \big((Pa \land Py) \land a \neq y\big)$	for existential instantiation
3	$\mid\ \mid\ \big((Pa \land Pb) \land a \neq b\big)$	for existential instantiation
4	$\mid\ \mid\ Pa$	$\land$E 3
5	$\mid\ \mid\ \exists x\, Px$	$\exists$I 4
6	$\mid\ \exists x\, Px$	$\exists$E 2, 3–5
7	$\mid\ \exists x\, Px$	$\exists$E 1, 2–6

2.

1	$\mid \exists x\big[Px \land \big(\forall y\,(Py \to y = x) \land Qx\big)\big]$	
2	$\mid\ \mid \big[Pa \land \big(\forall y\,(Py \to y = a) \land Qa\big)\big]$	for existential instantiation
3	$\mid\ \mid Pa$	∧E 2
4	$\mid\ \mid \big(\forall y(Py \to y = a) \land Qa\big)$	∧E 2
5	$\mid\ \mid \forall y(Py \to y = a)$	∧E 4
6	$\mid\ \mid\ \mid (Pb \land Pc)$	for conditional proof
7	$\mid\ \mid\ \mid\ \mid b \neq c$	for *reductio*
8	$\mid\ \mid\ \mid\ \mid (Pb \to b = a)$	∀E 5
9	$\mid\ \mid\ \mid\ \mid (Pc \to c = a)$	∀E 5
10	$\mid\ \mid\ \mid\ \mid Pb$	∧E 6
11	$\mid\ \mid\ \mid\ \mid Pc$	∧E 6
12	$\mid\ \mid\ \mid\ \mid b = a$	→E 8, 10
13	$\mid\ \mid\ \mid\ \mid c = a$	→E 9, 11
14	$\mid\ \mid\ \mid\ \mid c = c$	=I
15	$\mid\ \mid\ \mid\ \mid a = c$	=E 13, 14
16	$\mid\ \mid\ \mid\ \mid b = c$	=E 12, 15
17	$\mid\ \mid\ \mid\ \mid b \neq c$	R 7
18	$\mid\ \mid\ \mid b = c$	¬E 7–17
19	$\mid\ \mid \big((Pb \land Pc) \to b = c\big)$	→I 6–18
20	$\mid\ \mid \forall y\big((Pb \land Py) \to b = y\big)$	∀I 19
21	$\mid\ \mid \forall x\,\forall y\big((Px \land Py) \to x = y\big)$	∀I 20
22	$\mid \forall x\,\forall y\big((Px \land Py) \to x = y\big)$	∃E 1, 2–21

3.

1	$\forall x\, \forall y \big(\neg(Tx \land Ty) \lor x = y\big)$	
2	$\exists y \big((Ta \land Ty) \land a \neq y\big)$	for *reductio*
3	$\big((Ta \land Tb) \land a \neq b\big)$	for existential instantiation
4	$\exists y \big((Ta \land Ty) \land a \neq y\big)$	for *reductio*
5	$\forall y \big(\neg(Ta \land Ty) \lor a = y\big)$	∀E 1
6	$\big(\neg(Ta \land Tb) \lor a = b\big)$	∀E 5
7	$a \neq b$	∧E 3
8	$\neg(Ta \land Tb)$	∨E 6, 7
9	$(Ta \land Tb)$	∧E 3
10	$\neg\exists y \big((Ta \land Ty) \land a \neq y\big)$	¬I 4–9
11	$\neg\exists y \big((Ta \land Ty) \land a \neq y\big)$	∃E 2, 3–10
12	$\exists y \big((Ta \land Ty) \land a \neq y\big)$	R 2
13	$\neg\exists y \big((Ta \land Ty) \land a \neq y\big)$	¬I 2–12
14	$\forall x\, \neg\exists y \big((Tx \land Ty) \land x \neq y\big)$	∀I 13

4.

1	$\vert\ \forall x\,\forall y\big((Ax \land Ay) \to x = y\big)$	
2	$\vert\ \vert\ \exists x\,\exists y\big((Ax \land Ay) \land x \neq y\big)$	for *reductio*
3	$\vert\ \vert\ \vert\ \exists y\big((Ab \land Ay) \land b \neq y\big)$	for existential instantiation
4	$\vert\ \vert\ \vert\ \vert\ \big((Ab \land Ac) \land b \neq c\big)$	for existential instantiation
5	$\vert\ \vert\ \vert\ \vert\ \vert\ \exists x\,\exists y\big((Ax \land Ay) \land x \neq y\big)$	for *reductio*
6	$\vert\ \vert\ \vert\ \vert\ \vert\ \forall y\big((Ab \land Ay) \to b = y)\big)$	∀E 1
7	$\vert\ \vert\ \vert\ \vert\ \vert\ \big((Ab \land Ac) \to b = c\big)$	∀E 6
8	$\vert\ \vert\ \vert\ \vert\ \vert\ Ab$	∧E 4
9	$\vert\ \vert\ \vert\ \vert\ \vert\ Ac$	∧E 4
10	$\vert\ \vert\ \vert\ \vert\ \vert\ (Ab \land Ac)$	∧I 8, 9
11	$\vert\ \vert\ \vert\ \vert\ \vert\ b = c$	→E 7, 10
12	$\vert\ \vert\ \vert\ \vert\ \vert\ b \neq c$	∧E 4
13	$\vert\ \vert\ \vert\ \vert\ \neg\exists x\,\exists y\big((Ax \land Ay) \land x \neq y\big)$	¬I 5–12
14	$\vert\ \vert\ \vert\ \neg\exists x\,\exists y\big((Ax \land Ay) \land x \neq y\big)$	∃E 3, 4–13
15	$\vert\ \vert\ \neg\exists x\,\exists y\big((Ax \land Ay) \land x \neq y\big)$	∃E 2, 3–14
16	$\vert\ \vert\ \exists x\,\exists y\big((Ax \land Ay) \land x \neq y\big)$	R 2
17	$\vert\ \neg\exists x\,\exists y\big((Ax \land Ay) \land x \neq y\big)$	¬I 2–17

5.

1	$\vert\ \underline{\exists x \big(Tx \land \forall y\, (Ty \to y = x)\big)}$	
2	$\vert\ \vert\ \underline{\exists x\, \exists y \big((Tx \land Ty) \land x \neq y\big)}$	for *reductio*
3	$\vert\ \vert\ \vert\ \underline{\exists y \big((Ta \land Ty) \land a \neq y\big)}$	for existential instantiation
4	$\vert\ \vert\ \vert\ \vert\ \underline{\big((Ta \land Tb) \land a \neq b\big)}$	for existential instantiation
5	$\vert\ \vert\ \vert\ \vert\ \vert\ \underline{(Tc \land \forall y \big(Ty \to y = c)\big)}$	for existential instantiation
6	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ \underline{\exists x\, \exists y \big((Tx \land Ty) \land x \neq y\big)}$	for *reductio*
7	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ \forall y\, (Ty \to y = c)$	∧E 5
8	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ (Ta \land Tb)$	∧E 4
9	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ (Ta \to a = c)$	∀E 7
10	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ (Tb \to b = c)$	∀E 7
11	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ Ta$	∧E 8
12	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ Tb$	∧E 8
13	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ a = c$	→E 9, 11
14	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ b = c$	→E 10, 12
15	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ b = b$	=I
16	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ c = b$	=E 14, 15
17	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ a = b$	=E 13, 16
18	$\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ a \neq b$	∧E 4
19	$\vert\ \vert\ \vert\ \vert\ \vert\ \neg\exists x\, \exists y \big((Tx \land Ty) \land x \neq y\big)$	¬I 6–18
20	$\vert\ \vert\ \vert\ \vert\ \neg\exists x\, \exists y \big((Tx \land Ty) \land x \neq y\big)$	∃E 1, 5–19
21	$\vert\ \vert\ \vert\ \neg\exists x\, \exists y \big((Tx \land Ty) \land x \neq y\big)$	∃E 3, 4–20
22	$\vert\ \vert\ \neg\exists x\, \exists y \big((Tx \land Ty) \land x \neq y\big)$	∃E 2, 3–21
23	$\vert\ \vert\ \exists x\, \exists y \big((Tx \land Ty) \land x \neq y\big)$	R 2
24	$\vert\ \neg\exists x\, \exists y \big((Tx \land Ty) \land x \neq y\big)$	¬I 2–23

In the Introduction to his volume *Symbolic Logic*, Charles Lutwidge Dodson advised: “When you come to any passage you don’t understand, *read it again*: if you *still* don’t understand it, *read it again*: if you fail, even after *three* readings, very likely your brain is getting a little tired. In that case, put the book away, and take to other occupations, and next day, when you come to it fresh, you will very likely find that it is *quite* easy.”

The same might be said for this volume, although readers are forgiven if they take a break for snacks after *two* readings.

about the authors:

P.D. Magnus is a professor of philosophy in Albany, New York. His primary research is in the philosophy of science.

Thomas Donaldson is an associate professor of philosophy at Simon Fraser University. His primary areas of research are in metaphysics, epistemology, and the philosophy of mathematics.

Bruno Guindon is a senior lecturer of philosophy at Simon Fraser University. His primary areas of research are in practical and epistemic normativity.